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NOTES ON THE THEORY OF ELECTRONS

William Duane

According to the Faraday-Maxwell Theory, the following laws are true for the ether:

1st. The line integral of the electric force, or the E. M. F. around any closed curve, fixed in space, equals the rate at which the lines of magnetic force through any surface bounded by the curve *decrease*, the positive direction of line integration, and the positive direction of the lines of force through the surface being related to each other as the motion of rotation and translation of a right-handed screw.

2d. The line integral of the magnetic force, or the magneto-motive force around any closed curve, fixed in space, equals the rate of increase of the number of lines of electric force through any surface bounded by the curve.

The first law is a generalization from Faraday's law of induced currents, and applies, even if the change in the number of lines of magnetic force is due to the motion of magnets, provided the surface considered does not pass through magnetic matter. We may remove the last provision if we assume that magnetization in magnetic matter is due to the orientation of those small elementary magnets that are assumed on the Ampere-Weber hypothesis to constitute a magnet, for in this case the lines of induction consist of the lines of force of the field plus the lines of force due to said orientation. Adopting this hypothesis, the first law is perfectly general, and applies to the cases where the changes in the magnetic field are due to the motion of translation and rotation of the elementary magnets, as well as to those where the changes are due to the inductive effects of the electric force.

It is easily proved that under these conditions the total number of lines of magnetic force passing outward through a closed surface cannot vary with the time (Fig. 1). The e. m. f. around the closed line l, equals the rate of decrease of the lines of magnetic force through the surfaces S_1 and S_2 from left to right. The rate of decrease in the number of lines is the same for S_1 as for S_2, and equals the rate of increase in the number of lines through S_1 from right to left—i. e., outward from the space enclosed by S_1 and S_2. Hence, considering S_1 and S_2 as a single closed surface, the rate of decrease in the number of lines passing outward through it equals the rate of its increase; or the total number of lines of magnetic force passing outward from any closed surface does not vary.

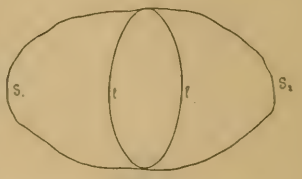

Fig. 1.

Apply the above to a small closed surface fixed in space surrounding one and only one of the elementary magnets. The number of lines of force outward through the surface cannot vary, even if the magnet should move completely out of the enclosure. Hence there must be as many lines of force running toward the magnet as away from it.

The question now arises: If we adopt the electronic theory of the conduction of electricity, can we generalize the second law so as to hold for material bodies, even in case the change is due to motion of the electrons? It is immediately evident that we cannot do so without special hypotheses, for if we surround an electron the charge of which is e, by a small closed surface, the number of lines of electric force through the surface outward is $4\pi e$, and if the electron moves out of the enclosure the number changes to zero.

We may, however, generalize the second law as follows: The magneto-motive force around any closed curve equals the rate of increase of the number of lines of electric force running through any surface bounded by the curve, provided we take account of that inincrease only which is due to the cutting of lines of force across the closed curve.

In the case of a magneto-motive force produced by a conduction current of electricity, this cutting of lines of electric force across the closed curve is due primarily to the motion of the electrons (at least according to the modern views). Let us consider a constant current of electrons flowing in the positive direction around a circuit through the line l in Fig. 1. As each electron with its charge flows around the circuit once, the entire number of its lines of force $4\pi e$ cut across the line l. At some instant during its motion, the electron passes through the surface S_1 and, the number of its lines running through this surface in the positive direction, decreases by the amount $4\pi e$. Hence, on the whole, the change in the number of lines of force through this surface, due to the motion of this electron, is zero. The current has been assumed constant. Hence, as a whole, the number of lines passing through S_1 remains the same; which means that as a whole just as many lines of force are withdrawn from the positive direction through the surface, owing to the passage of electrons through S_1 as are thrust through by cutting the edge of the surface. But the magneto-motive force equals the number of lines of force cutting through l per second, and therefore equals 4π times the sum of the electron charges passing through S_1 per second. This latter we consider to be the current through S_1. The same is true of any other surface, S_2 bounded by l, and hence the sum of the electrons passing through S_1 per second, is the same as through S_2, or the total sum of the charges passing outward through the closed surface S_1 S_2 per second is zero. This means that the flow of electrons is similar to that of an incompressible fluid.

Let us consider now the case of a variable current. We have very strong reasons for believing that in this case, too, the magneto-motive force around any closed line in space equals 4π times the total current through any closed surface bounded by the line, but in making up the total current we must add to the conduction current that which Maxwell called the displacement current. This means that the total current into any closed space equals the total current out of it, or that even in the variable state a current of electricity fills space exactly as if it were the flow of an incompressible fluid.

Although we cannot say that the above is a logical proof, yet we may say it suggests the hypothesis that a displacement current of electricity is the motion of electrons just as a conduction current is. In fact such a hypothesis leads to a very interesting mental picture of Maxwell's dielectric displacement of electricity, such displacement being nothing more nor less than the actual displacement of the electrons in the dielectric.

To fix our ideas let us take a particular case. Suppose that we have a condenser K, and that we charge this by joining up a cell, as in Fig. 2. We will suppose that there are pairs of charged electrons in the dielectric between the plates. During the flow of the current into the plates, a number of + charged electrons pass into A. Owing to their forces of attraction and repulsion they tend to push the positive electrons of each pair in the dielectric from the plate A toward B, and to pull the negative from B toward A. Thus while the charging lasts there is an actual motion of positively charged electrons from A to B and negatively charged electrons from B to A, which motion produces exactly the same magnetic effect as the equivalent conduction current would produce, and is a real current of electricity. There are, however, very few dissociated electrons in the dielectric; that is, very few electrons that are free to move completely across, and carry the current conductively from A to B. The result is, therefore, that the charging will continue only until the electrons have been forced so far from the original positions of equilibrium that their mutual attractions can withstand the electro-motive force of the cell, and prevent any further motion. Under these circumstances the condenser is charged. The positive electron of each pair has been pushed toward B and the negative toward A; in other words, the dielectric is polarized.

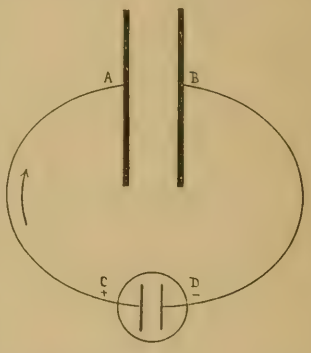

Fig. 2.

This leads to interesting conceptions of Faraday's tension and pressure along and perpendicular to the lines of polarization. Owing to the motion of the electrons, the positive electron of one pair has come nearer the next pair to the right than it originally was, and the negative electron of the second pair has approached nearer the first pair than it originally was. Hence there will be a greater attraction between the two electrons than before; in other words, there is an increased tension between the two pairs of electrons. The same will be true of the second pair of electrons and the pair next to the right of it, etc. Hence there will be lines of tension between successive pairs of electrons across from plate A to plate B, and hence a force pulling the electrons in the two plates, and therefore the two plates themselves together.

It is easily seen, too, that in general there will be an increased repulsive force between two pairs of electrons lying one above the other, for if the positive electrons in each pair are pushed so as to point in one direction, and the negative so as to point in the other, the positive electron in one pair will in general be a little further away from the negative in the other than before, and hence the attraction between them will be diminished, which means, since they were in equilibrium before, that there is now a resulting repulsion between the two pairs. The same being true of all pairs lying one above the other or one alongside of the other not in the line of polarization, there is a resulting pressure perpendicular to that line of polarization.

Applying the law that the total electric current into a closed space must equal that out of it, to the surface of plate A, we see that although the plate is charged, the algebraic sum of the electron charges must be the same as before. The same is true of plate B. In fact, according to the above theory, we must mean by the charge on a conductor simply the polarization or displacement of electrons in the surrounding dielectric. The conductor is charged positively if the positive electrons in the dielectric are displaced away from the conductor and the negative toward it; and, negatively, if the positive electrons are displaced toward the plate and the negative away from

it. This view is very similar to Maxwell's idea of the charge on a conductor. Maxwell, however, apparently considered electricity as a single incompressible fluid, that could be displaced either in one direction or the other at any single point, but not in both directions at once, whereas, according to the present point of view the electrons do not fill space entirely, and the positive electrons may be displaced in the opposite direction to the negative.

The mechanical force pulling the two plates (Fig. 2) together is not due to an excess of positive electrons in plate A attracting an excess of negative electrons in plate B, but for the most part to the displacement of electrons along the lines of polarization, and the consequent system of pressures and tensions. We cannot, however, do away with the ether on this hypothesis, for the ether is necessary in order to explain the attractions and repulsions of neighboring electrons, unless, of course, we postulate direct action at a distance between them. It is interesting to note, that if we accept the usually quoted values for the number of molecules in a cubic cm. of gas, and, the charge on an electron, the average extreme displacement of an electron before a spark passes through the gas is, roughly estimated, 3×10^{-10} cm. if we assume there are two electrons in each molecule. This displacement is $\frac{1}{300}$th of the average distance between the molecules.

Granting the hypothesis that a current of electricity is due to the motions of charged electrons, it follows that Weber's molecular currents must be nothing more nor less than the revolution of the two or more electrons in a molecule. If the positive and negative electrons had the same mass they would move with the same velocity around paths of the same size, and the magnetic effect of one would practically annul that of the other, since they must revolve in the same direction and a negative charge traveling in one direction means a current in the opposite direction. J. J. Thompson, however, has shown that the negative electron is about one thousand times smaller than the rest of the atom or molecule. Hence the negative electron will revolve with greater velocity around a larger circuit than the

rest of the atom; and the magnetic effect of the negative electron will greatly exceed that of the positive.

Of course the above idea as to the origin of the pressures and tensions in an electric field is applicable also to the magnetic field, the tension being due to the mutual attraction of the elementary magnets or revolving electrons along the line of the magnetic polarization, and the pressures to the mutual repulsions perpendicular to this line.

AN ELECTRICAL THERMOSTAT

WILLIAM DUANE AND CHARLES A. LORY

In some research work that one of us has been carrying on recently, it became necessary to construct an easily adjustable thermostat that would keep the temperature of a bath constant to within $\frac{1}{1000}$th of a degree Centigrade for a considerable length of time. It was thought that this could be accomplished best by means of an electric current, because if the current passed through wires suspended in the bath, or through a conducting bath itself, heat would be supplied throughout the whole bath much more easily and quickly than by other means.

The result of our endeavor to construct such an electrical thermostat has been quite satisfactory. During the trial runs the temperature of the thermostat remained constant to within less than $\frac{1}{2000}$th of a degree Centigrade, although several times the temperature of the surrounding atmosphere varied 12° C. or 15° C. in half an hour.

The general scheme is this. Through a conducting liquid, or through wires immersed in one which is non-conducting, flows an electric current, that is sufficiently large to heat the liquid up to a temperature considerably above the constant temperature required. A system of tubes containing a liquid with a large temperature-coefficient of expansion is placed in the bath. By means of a suitable mechanism the expansion of this liquid interrupts or reduces the strength of the heating current when the required temperature has been reached. The temperature of the bath then begins to fall, whereupon the original current is started again automatically. It

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might seem at first thought, that, owing to the time required for the heat to penetrate through the walls of the tubes to the expanding liquid within, the making and breaking of the circuit would take place rather slowly, and the temperature of the bath would be oscillatory instead of constant. This is undoubtedly true to a certain extent. Practically, however, with our arrangement of apparatus the variation of temperature is too small to be detected even by a differential thermometer that would indicate a change of temperature of $\frac{1}{2000}$th of a degree Centigrade. Indeed the efficiency of the apparatus is due to the fact that the makes and breaks follow each other so rapidly, that there is not time for the temperature to change perceptibly between them. Often two, three or even more interruptions of the current occur in one second. At first we used an ordinary wash-boiler containing about 150 liters of water as a bath. The boiler was placed in a large wooden box and packed in around the sides with wool. A light wooden frame immersed in the water served as a rack to hold the wires conveying the current. We encountered considerable difficulty, however, owing to the wires becoming very brittle and breaking after a few days' use. This was the case with wires of copper, iron and German silver. A direct current, too, seemed to produce a much greater effect than an alternating one of the same heating power.

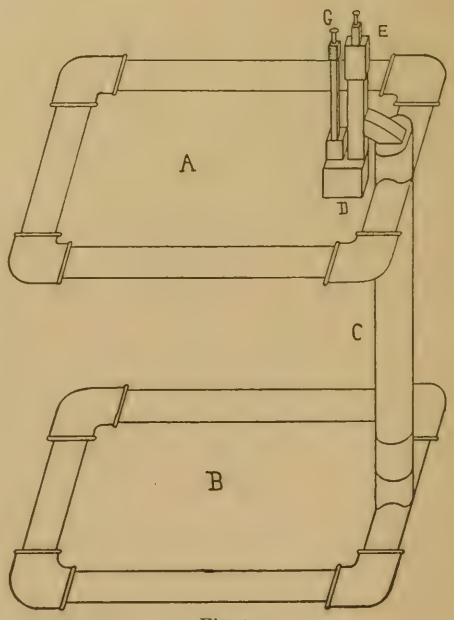

Fig. 1.

To obviate this difficulty we constructed a wooden trough ($80^{cm} \times 40^{cm} \times 50^{cm}$) and filled it with a weak solution of

common salt in water. The two ends of the trough were completely covered inside with sheets of zinc, that served as electrodes by means of which an alternating current from the 52-volt electric light circuit passed through the solution. This method of heating proved very satisfactory.

The liquid in the bath was kept in continual and rapid circulation by means of four stirrers operated by a small water motor.

The tubes containing the expanding liquid were of thin brass about 2·5^{cm} in diameter. They were fastened together by means of iron joints in the form of two rectangles (A and B, Fig. 1). The tube C connected the two rectangles together. The whole system of tubing was placed in the bath with the planes of the rectangles horizontal and the tube C vertical. The portion G D is the regulating device. After the tubes had been filled through a small hole in the top of the tube E a brass cap was screwed on the end of E, pressing a small disk of lead tight down over the hole. This formed a perfectly air-tight stopper. The form of the stopper and regulating device finally adopted is shown in Fig. 2. The portions of the tube marked *a*, *c* and *d* were of iron; *b* was of brass, and *e* of glass. *a* and the whole system of tubes A, B and C were filled with alcohol, and *b*, *c*, *d* and *e* up to the platinum wire at *f* with pure mercury. The alcohol in the large system of tubes expanding and contracting forced the mercury up and down, making and breaking the contact at *f*. At the joints between *a* and *b*, *b* and *c*, and *d* and *e* were rubber washers; and screw clamps, not shown in the figure, pressed the several parts firmly together.

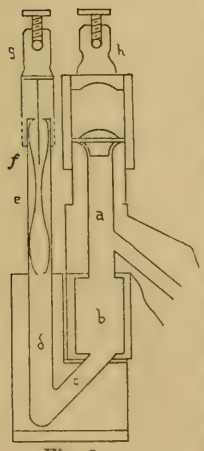

Fig. 2.

The reason for inserting the small brass reservoir *b* in the portion of the tube containing the mercury was this. Without it we found that the temperature at which contact was made and broken at *f* kept gradually rising, indicating a small leakage of alcohol. Since the insertion of an easily amalgamated metal in part of the tubing contain-

ing the mercury completely corrected this fault, we surmise that the leakage must have taken place along the contact surface between the mercury and the iron tubing. No appearance of alcohol at *f* was noticed, but it could easily have evaporated too rapidly.

The binding posts at *g* and *h* were connected to a circuit containing a dry cell and a relay. The relay opened and closed a second circuit containing an electromagnet, which in turn controlled the main heating circuit.

For temperatures only a few degrees above that of the room the makes and breaks at *f* may open and close the heating circuit completely; but for temperature 30° or 40° C. above that of the room it is better to arrange the circuits so that the makes and breaks at *f* throw an extra resistance into and out of the main circuit. In this case, of course, the larger of the two main currents must be sufficient to heat the bath up to a higher temperature than the required one, and the smaller insufficient to maintain it at that temperature.

In our apparatus the glass tube *e* was open at the top. A large variation in the atmospheric pressure might produce a change in the temperature at which the makes and breaks take place on account of the slight compressibility of the alcohol. No such effect has been noticed, however. If it occurred the fault could easily be remedied by fastening a tube with a large bulb on its end to the tube *e*. The bulb, of course, would have to be below the surface of the bath, so that the temperature of the air within it would not change.

In practice it is easy to set the regulating device to working at any desired temperature between that of the room and one a few degrees below the boiling point of the liquid in the tubes, as follows. With the cap *h* unscrewed, allow the bath to heat up slowly, and when the desired temperature has nearly been reached, screw the cap down. A little practice will enable one to set the thermometer at a temperature within a small fraction of a degree of the desired one.

The reason that the interruptions of the current follow each other so rapidly is not quite clear to us. It may be due to a slight jarring of the surface of the mercury at *f*. In any case the best results are obtained with pure dry mercury, and when the platinum wire touches the convex surface of the mercury column near its center.

Hale Physical Laboratory, University of Colorado, U. S. A.

ON THE VELOCITY OF CHEMICAL REACTIONS

WILLIAM DUANE

In investigating the laws of physical chemistry it is expedient for the physicist to devise and develop the methods of measurement, and for the chemist to apply them. In the following pages are described two methods of measuring the velocity of chemical reactions. The velocity of a chemical reaction is the rate at which a chemical compound appears or disappears during that reaction. The changes in the quantity of this compound present during successive intervals of time are measured by the changes in some property of the chemical system during the intervals. This property is usually either a chemical or a physical one.

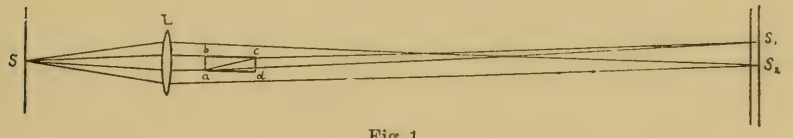

Fig. 1.

In the first of the following methods the basis of the measurement is the change in the index of refraction of the system, and in the second the change in its volume.

The first method is applicable to those chemical systems only that are transparent. It is substantially the following: Rays of light from an illuminated slit S (Fig. 1) passing through a long focus lens L and the tube *a b c d* form a distant image S_1 of S. The slit S is perpendicular to the plane of the diagram which represents a horizontal section of the apparatus. The tube *a b c d* has plane glass ends *a b* and *c d;* and a plane glass plate *a c* divides it into two

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wedge-shaped compartments. The ends *a b* and *c d* are not quite parallel to each other, so that if the two compartments are filled with liquids having the same index of refraction there will be a slight resultant refraction of the light rays that pass through the tube. The rays of light that pass outside of the tube, therefore, will form an image S_2 a little to one side of S_1. It is evident that if the liquid in one compartment (the wedge *a c d* for instance) is undergoing a chemical change its index of refraction in general will vary and the

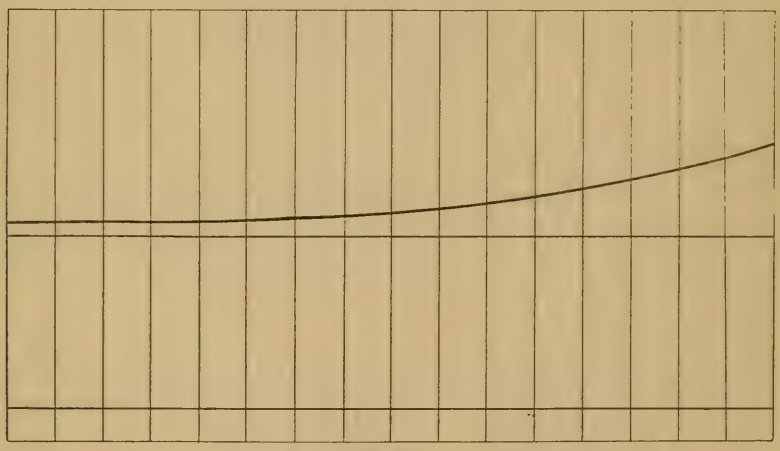

Fig. 2.

image S_1 will move sideways. The distance that S_1 has moved will be a measure of the change that has taken place in the index of refraction and, therefore, of the amount of substance in *a c d*, that has reacted. The displacements of S_1 can be determined by comparing its distances from S_2 which remains stationary.

In order to obtain a complete record of the position of S_1 a photographic plate S_1S_2 is placed in a vertical position at the images S_1 and S_2 and just in front of it a screen. A narrow horizontal slit cut in the screen allows a small part of the light only to pass through. At any instant of time, therefore, there will be two small spots of light on the plate at the intersections of the two images S_1 and S_2 with the projection of the slit on the plate. A system of cog-wheels

allows the photographic plate to fall slowly during the reaction so that two lines are drawn on it, one of them straight due to the fixed image S_2 and the other curved due to the moving image S_1. The curved line represents the reaction in that the absyssas are proportional to the intervals of time and the ordinates represent (but are not proportional to) the quantities of the substance that have reacted. After the reaction is completed the plate is drawn up and lowered again and the image S_1 traces a third line that is practically straight. This line may be taken as the zero line and the distances between it and the curved one are (at least in some cases) proportional to the amounts of the original compounds left in the solution.

Fig. 2 is a reduced copy of a photograph representing the inversion of a 25 per cent. solution of cane sugar, the inversion being accelerated by the addition of hydrochloric acid. The middle horizontal line represents the position of the image S_1 twenty-four hours after the reaction had started. The vertical lines were drawn with a dividing engine after the plate had been developed. The distance between two successive lines represents fifteen minutes. This distance was determined on a separate plate by lighting a magnesium burner for an instant every hour at some distance in front of the horizontal slit, and by measuring on the dividing engine the distance between the lines thus formed. A heavy verticle line to the extreme right of the plate (not seen in the copy) is a magnesium flash-light line and represents the instant at which the hydrochloric acid and sugar solution were mixed.

The photographic plate was fastened in a frame hung on a fine iron wire that was wrapped around a cylinder on the axle of one of the wheels in the works of a clock. The escapement of the clock was operated by an electro-magnet, the circuit of which was made and broken by the swing of the pendulum of a standard clock. On account of the escapement the downward motion of the plate was by jerks, but as each jerk carried the plate only about $\frac{1}{20}$th of a mm. this is not apparent from the photograph.

It is evident from the way in which the fifteen-minute lines are drawn that no correction need be made for the shrinkage of the

gelatine films, for practically the same shrinkage takes place on one plate as on another. The shrinkage, too, reduces all the ordinates of the curve in the same proportion, and if the relative positions of the images are always determined by measurements on a photographic plate no correction need be applied to the ordinates.

The object of having two wedge-shaped compartments is to reduce the dispersion of the light passing through them as much as possible, indeed white light from an incandescent lamp can be used. The solution in the wedge $a\ b\ d$ is the same as that in $a\ c\ d$ except that in it the reaction has already taken place. Under these circumstances the indices of refraction in the two wedges are very nearly equal and a slow change of a few degrees in temperature does not displace the image S_1 since such a change affects the index of refraction in both solutions practically to the same extent.

In the above described experiment the dimensions of the apparatus were as follows: SL (Fig. 1) $= 150^{cm}$; $LS_2 = 250^{cm}$; $ab = 1^{cm}$; $ad = 5{\cdot}3^{cm}$; $Lb = 15^{cm}$, approximately. The breadth of the slit in front of the figure was $\cdot 2^{mm}$, and its distance from the figure about $\cdot 3^{cm}$.

That the distance of the image S_1 at any time from its final position after the reaction has been completed is very approximately proportional to the amount of cane sugar left in the solution at that time, may be seen from the following reasoning: The proof is based upon the assumptions that the density d and the specific index of refraction $\frac{n^2-1}{n^2+2} \cdot \frac{1}{d}$ of the solution (where n is the ordinary index of refraction) are additive functions of the constituents (see Nernst, *Theoretische Chemie*). Granting these, it follows that the difference between the value of $\frac{n^2-1}{n^2+2}$ at any time and its final value must be proportional to the amount z of cane sugar remaining at that time. Denoting this difference by $\triangle$ we have

$$\triangle \frac{n^2-1}{n^2+1} = k\,z$$

Approximately for small changes

$$\triangle\frac{n^2-1}{n^2+2}=\frac{d}{dn}\left(\frac{n^2-1}{n^2+2}\right)\triangle n=\frac{6n}{(n^2+2)^2}\triangle n$$

Hence

$$\triangle n=\frac{(n^2+2)^2}{6n}\,k\,z$$

If n (Fig. 3) is the index of refraction of the already reacted solution

$$\frac{n_1}{n}=\frac{\sin\beta}{\sin a}=\frac{\sin(a+\triangle a)}{\sin a}$$

$$=1-\cot a\,\sin\,\triangle a$$

approximately, and therefore,

$$\cot a\,\sin\,\triangle a=1-\frac{n_1}{n}=\frac{\triangle n}{n}$$

or

$$\sin\,\triangle a=\frac{\triangle n}{n\cot a}$$

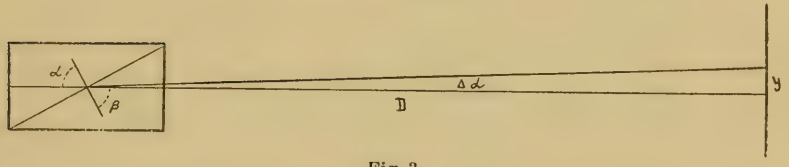

Fig. 3.

If D is the distance from the solution to the screen

$$\frac{y}{\mathrm{D}}=\frac{\sin\,\triangle a}{\cos\,\triangle a}.$$

$$\therefore y=\mathrm{D}\,\sin\,\triangle a \text{ approximately}$$

Hence

$$y=\frac{\mathrm{D}(n^2+2)^2}{n^2\,\cot a}\,k\,z$$

n does not vary more than ·1 per cent.; and if n lies between 1·3 and 1·4, which it usually does, the expression $\frac{(n^2+2)^2}{n^2}$ is practically constant.

Hence y is proportional to z, or the displacement of the image from its final position is proportional to the quantity of cane sugar in the solution.

Whether or not y is proportional to z in any particular case should and can be tested experimentally as follows: Take equal

quantities by weight of a solution that has already reacted and of one that is just beginning to react. Mix them together and obtain a curve as above. This mixture is equivalent to a solution that has reacted half way, and the total change in the position of S_1 as indicated by the curve should be $\frac{1}{2}$ that in the case of the original solution. Such a curve photographed for the inversion of sugar proved the proportionality to within about $\frac{1}{2}$ of a per cent. If y is found not to be proportional to z a number of mixtures must be made with varying quantities of new and old solutions, and the relation between y and z determined.

The distances between the lines on the photographic plate corresponding to the different instants of time can be measured by means of a micrometer microscope of low power. I have found it more satisfactory, however, to place a glass scale over the plate and take the readings with a small lens, illuminating both scale and plate by means of a mirror below them.

The following table contains the results of such measurements on the plate representing the inversion of sugar. The first column contains the time t expressed in minutes; the second, the distances y in centimeters between the curve and the line drawn 24 hours later (the value of y for $t=0$ being extrapolated), and the third, the percentage of cane sugar left in the solution:

t	y	% sugar	$\frac{k}{2{\cdot}3026}$
0	25·5	25·	·00535
15	21·2	20·78	
30	17·9	17·54	·00490
45	15·4	15·09	·00463
60	13·25	12·99	·00454
75	11·3	11·07	·00455
70	9·75	9·56	·00449
105	8·5	8·33	·00441
120	7·4	7·25	·00435
135	6·4	6·27	·00433
150	5·55	5·44	·00431
165	4·85	4·75	·00427
180	4·3	4·22	·00419
195	3·95	3·87	·00405
210	3·6	3·53	·00395
225	3·3	3·23	·00385

According to Gouldberg and Waage's law the rate at which the cane sugar disappears should be proportional to the amount left in the solution. Hence $\frac{dz}{dt} = -k\,z$.

Or integrating between t_1 and t

$$k\,(t-t_1) = \log_e \frac{z_1}{z} = 2{\cdot}3026 \log_{10} \frac{z_1}{z}$$

Or $$k = \frac{2{\cdot}3026}{t-t_1} \log_{10} \frac{z_1}{z}$$

The third column in the table contains the values of $\frac{k}{2{\cdot}3026}$ calculated by this formula, using the value $t_1 = 15$ minutes so as to avoid extrapolation. It appears that k as calculated above is not a constant.

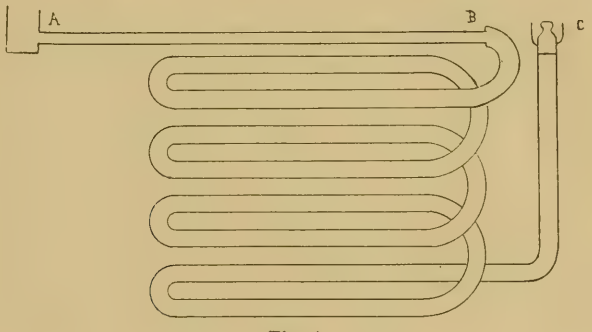

Fig. 4.

The decrease in its value indicates that the velocity of the reaction is greater at the beginning or less at the end than would be expected from Gouldberg and Waage's law. During the reaction, of course, the amount of water present changes perceptibly, but not enough to account for the large variation of k. The total change during the entire reaction is less than 2 per cent. The deviation from Gouldberg and Waage's law may be due to the relatively large quantity of hydrochloric acid in the solution. To decide whether it is or not will be the object of further investigation.

The second method is based upon the change in the volume of the chemical system during the reaction. In order to measure the

change in volume the system is placed in a spiral glass tube (Fig. 4) on one end of which is sealed a capillary tube A B, and on the other end a ground stopper and cup C. The reacting system may fill the entire spiral or only a part of it, the rest being filled with a liquid that does not combine with the given system. If the volume increases during the reaction, the cup A and most of the tube A B are filled with mercury. If the volume decreases, the cup and only a small part of the tube are filled. Evidently during the change of volume the mercury is drawn in or pushed out through the capillary tube and the positions of the end of the mercury column measure the amount of change.

The particular form represented (Fig. 4) was adopted, firstly, in order that the temperature of the system should remain that of a bath in which the spiral is immersed owing to the large surface presented to the bath; and secondly, in order that the tube might be filled, and the first reading of the position of the mercury column taken as soon after the reaction started as possible. The liquid is first poured in at C and allowed to fill the spiral, the tube A B and the cup A. Mercury is then poured into A and finally the stopper inserted at C. A little practice in manipulating the stopper and tilting the whole tube will enable one to set the dividing surface between the mercury and the other liquid at any desired point in the capillary tube. The tube A B is graduated or a graduated scale is fastened to it.

In most cases the change in the volume of a system is small. Hence, if the temperature of the apparatus varies during the reaction, even by a small amount, the consequent change in volume will be a large fraction, perhaps, of the change we wish to measure. It is, therefore, necessary to immerse the spiral tube in a delicate thermostat. In the following experiment on the inversion of a 25 per cent. solution of cane sugar the electrical thermostat described by my assistant, Mr. Lory, and myself in this Journal for March, 1900, was used. The temperature of the bath did not vary as much as $\frac{1}{2000}$th of a degree except once, and then the variation was only about $\frac{1}{1000}$th of a degree.

In this experiment the spiral tube contained about 100^{cc} of the solution. The length of the capillary tube A B was 36^{cm}, and its cross section about 1$^{sq\,mm}$. The temperature was 25°·872 C.

The first two columns of the following table contain the readings of the end of the mercury column, on a metal scale attached to the capillary tube, together with the corresponding intervals of time t in minutes.

After allowing the apparatus to stand 24 hours and the reaction to become practically completed, the bath was heated up again to 25°·872 C., allowing all of the mercury which had been drawn into the spiral to run out. The end of the mercury column then stood at the mark ·20, which is the zero corresponding to zero quantity of cane sugar in the solution. The third column in the table contains the readings referred to this zero.

Under the assumption that the volume of the solution is an additive function of the constituents, these latter distances representing the changes in volume should be proportional to the quantities of cane sugar left in solution. The last column contains the values of $\frac{k}{2{\cdot}3026}$ calculated from the formula

$$\frac{k}{2{\cdot}3026} = \frac{1}{t - t_1} \log_{10} \frac{z_1}{z}$$

t	cm.	cm.	$\frac{k}{2.3026}$
0			
20	34·55	34·35	
41	29·66	29·46	·00318
55	26·87	26·67	·00314
70	24·28	24·08	·00308
85	21·61	21·41	·00316
100	19·83	19·63	·00303
115	17·86	17·66	·00304
130	16·12	15·92	·00303
145	14·99	14·79	·00292
160	13·80	13·60	·00287
175	12·70	12·50	·00284
190	11·85	11·65	·00276
205	10·96	10·76	·00272
220	10·63	10·43	·00259
235	9·92	9·72	·00255
250	9·37	9·17	·00249
265	8·80	8·60	·00245

We see that k decreases, indicating as before that the velocity is less rapid at the end than would be expected from Gouldberg and Waag's law.

If desired, a complete record of the position of the mercury column could be obtained by any of the methods for recording the readings of a mercury thermometer.

The chief advantages of the above described methods are, first, that the chemical system is not disturbed during the reaction; second, that no time is lost in titration or other chemical tests for the state of the system, thus making it possible to investigate rapid reactions; third, that it is not necessary for the experimenter to make observations during the reaction, which is a tedious piece of work if the velocity is very small; and fourth, the curves are drawn by purely mechanical processes, thus avoiding the influence of the personal equation.

A third method based upon the change in the index of refraction of the chemical system as measured by the motion of interference bands is being perfected.

In conclusion I wish to thank my assistant, Mr. F. C. Blake, for the care with which he has helped me obtain the above data. He and his brother, Mr. J. C. Blake, are at work applying the methods to hitherto uninvestigated reactions.

Hale Physical Laboratory, University of Colorado.

ON THE CONGRUENCES OF TWISTED CURVES*

ARNOLD EMCH

1. Two surfaces with the equations

$$\begin{aligned} F\ (x, y, z, a, b)&=0, \\ \phi\ (x, y, z, a, b)&=0, \end{aligned} \tag{1}$$

where a and b are arbitrary parameters, define a *congruence of curves*. Any relation between a and b,

$$b=f\ (a) \tag{2}$$

gives rise to a system of curves which generate a *surface of the congruence*. The condition that among these surfaces there shall be surfaces being the envelope of its generating curves is

$$\frac{\delta F}{\delta a}+\frac{\delta F}{\delta b}\frac{db}{da}=0, \frac{\delta \phi}{\delta a}+\frac{\delta \phi}{\delta b}\frac{db}{da}=0. \tag{3}$$

Eliminating x, y, z, between (1) and (3), this condition may be expressed by a relation of the form

$$\psi\left(a, b, \frac{db}{da}\right)=0, \tag{4}$$

i. e., by a differential equation of the first order between the parameters a and b (1). Every solution of this equation is equivalent with a certain surface of the congruence having an envelope.

2. I shall now assume that (4) has a singular solution, so that the result of the elimination of p between

$$\left.\begin{aligned} \psi\ (a, b, p\ )&=0, \\ \frac{\delta \psi}{\delta p}&=0, \end{aligned}\right\} \tag{5}$$

and

satisfies (4). In this case the theorem holds:

* Presented at the 1900, December, meeting of the American Mathematical Society (Chicago Section).

(1) See Darboux, Théorie Générale des Surfaces, Vol. II, p. 7.

All surfaces defined by (4) *are tangent to the surface corresponding to the singular solution of* (4).

To prove this, let z_1 be a fixed value of z, while x and y are independent variables, and write

$$x=f_1(a, b, z_1), \qquad (6)$$
$$y=\phi_1(a, b, z_1),$$

in accordance with (1). Equations (6) represent a contact-transformation in the plane $z=z_1$, when a and b are interpreted as Cartesian co-ordinates in this plane, in the same manner as x and y(1). A line-element is transformed into another line-element, a system of line-elements into a system of line-elements (Elementverein), two curves in contact into two curves in contact of the same order. To every solution of (4) belongs a curve C with an equation of the form $g(a, b)=0$, which is tangent to the curve S representing the singular solution (5). Applying the contact-transformation (6) to the curves C and S, a new system of curves C' is obtained which are all tangent to the transformed singular curve S'. Now, to every solution $g(a, b)=0$ of (4), by (1), corresponds a surface of the congruence having an envelope, and it is evident that its trace on the plane $z=z_1$ is the curve C'. From this it is further seen that system of surfaces with an envelope contained in (1) intersect the plane $z=z_1$ in a system of curves C' having S' as an envelope. The same holds true for all values of z_1. As every value of z defines a curve S' it follows that the system of curves S' forms a surface which, according to (3) and (4) has an envelope. The system of surfaces of the congruence having an envelope is therefore tangent to a surface with an envelope, and corresponding to the singular solution of (4), as was to be proved.

3. Without entering into further details I shall apply the previous result to the case of a congruence of straight lines intersecting singly each of two given plane curves in space. As is well-known, the two parts of the focal surface of the congruence are the curves themselves, and the system of developable surfaces consists of

(1) See Sophus Lie, *Geometrie der Berührungs-Transformationen*, Vol. I, p. 10 and pp. 43-67.

all cones having their vertices on one of the given curves and passing through the other curves. In case that equation (4) has a singular solution, the system of cones are tangent to a developable surface D containing the given curves. Every tangent plane to D is also tangent to the two given curves, so that the singular developable surface may also be considered as generated by all tangent planes common to both of the given curves. If m and n are the classes of the given curves, the developable surface of its common tangent planes is of the class $m\,n$. Of particular interest is the case where one of the given curves is an infinitely distant circle (given by a cone-director) so that the generating planes of the developable surface all have the same inclination towards a fixed plane (perpendicular to the axis of cone-director.) Assuming the plane XOY parallel to the plane of the infinite circle, this problem is equivalent with a particular solution of Monge's equation

$$dx^2+dy^2-k^2dz^2=0, \quad (k\leqslant 0) \tag{7}$$

or of the partial differential equation

$$1+\left(\frac{\delta z}{\delta x}\right)^2+\left(\frac{\delta z}{\delta y}\right)^2=\frac{k^2+1}{k^2}, \tag{8}$$

where $\frac{1}{k}=tan\gamma$ is the tangent of the constant angle of inclination of the generating planes with the plane XOY(1). The class of the developable surface is now $2m$ and its order or rank

$$V=2m(2m-1)-2h-3\beta, \tag{9}$$

where h is the number of double tangents in a plane section of the developable surface and β the number of stationary planes of the latter.(2)

4. I shall now consider the curve of intersections, S, of the developable surface with the plane XOY, whose order is given by (9). The developable surfaces of the congruence are right cones whose elements include constant angles with (XOY) and whose base-circles

(1) Lie, *loc. cit.* p. 262.
(2) Fiedler, *Geometrie der Lage*, Vol. II, pp. 132-142.

envelope the curve S. The centers of these circles are situated on the orthographic projection upon (XOY) of the curve of class m, and these circles all intersect the trace of the plane containing the curve of class m on the XOY-plane under a constant angle. Hence the theorem: *The envelope of all circles whose centers are situated on a curve of class m and which cut a fixed straight line in their plane at a constant angle is a curve of class 2m.*

If $m=2$, i. e., in case of a conic, the class of the envelope will be 4 and its order generally 8, since it generally has two double-tangents. In case of 3 double tangents the order reduces to 6. It is an easy matter to locate the conic and the cone-director of the infinite circle in such a manner, that these and many other configurations of circular systems may be obtained. From the specializations of the general result under articles *3* and *4* it is seen that they involve the whole theory of *Cyclography*, or the representation of points in space by the circles of a plane and conversely, a method inaugurated principally by Fiedler (1) and rigorously treated by Lie, loc. cit.

University of Colorado, December 24, 1900.

(1) *Cyclographie*, Leipzig, 1882.

CYCLOGRAPHIC TRANSFORMATION OF ORDINARY SPACE

ARNOLD EMCH

1. INTRODUCTION.

In my foregoing paper on general congruences of curves in space, I have referred to a particular case of congruences of straight lines which result from the partial differential equation.

$$1+\left[\frac{\delta z}{\delta x}\right]^2+\left[\frac{\delta z}{\delta y}\right]^2=\frac{k^2+1}{k^2},$$

when x, y, z are the co-ordinates of a point describing a curve in space.[1]

These congruences have the closest connection with the geometry of the circle in a plane, and with the method of cyclography as it has been established by Steiner, Fiedler and others. This method admits of beautiful applications, and as it seems to be but little known, I shall attempt to present its principal features in an elementary manner,[2] and show its relations with more advanced theories.

[1] S. Lie, *loc. cit.*

[2] In 1826, J. Steiner announced that he had a manuscript, "über das Schneiden (mit Einschluss der Berührung) der Kreise in der Ebene, das Schneiden der Kugeln im Raume und das Schneiden der Kreise auf der Kugelfläche," ready for print. As W. Fiedler remarks, this paper is not in his collected works and must, therefore, have been lost. The foundations of this method were laid by *Cousinery*, who, in 1828, (Paris) published his *Géométrie perspective*. He introduced the *cercle à distance*, which plays such an important part as *Distanzkreis* in Fiedler's early investigations, and solved by this means Apollonius' problem. The establishment of this method as an independent geometrical branch, however, is due to Professor W. Fiedler, of Zürich, who, in 1880-82 published two memoirs

2. Point, Straight Line and Plane.

The points of space of three dimensions may be conveniently determined by the circles of a fixed plane in this space, if the center of every circle is considered as the orthographic projection of a point in space upon this plane, and its radius as the distance of the point from the plane. In this manner every point in space is determined by a certain circle in the plane. It is, however, evident that every circle in the plane represents two points on opposite sides of the plane (equi-distant from, and in the same perpendicular to, the plane). To distinguish two points of this kind by their representative circles, we may state that all circles described in the same sense represent points on one side of the plane, and all circles described in a contrary sense points on the other side of the plane. For graphical representation we assume the plane of reference as coincident with the plane of the drawing. A circle described counter-clockwise shall represent a point above, and one described clockwise a point below the plane. Graphically this may be indicated by arrows on the circles.

A straight line in a general position in space intersects the plane of reference in a point O. The centers of the circles representing such a line are situated on the orthographic projection of this line upon the plane, and the radï of these circles are proportional to the distances of their respective centers from the origin O. Conversely, all circles whose centers are situated on a straight line and whose radï are proportional to the distances of their respective centers from a fixed point on the given straight line, represent a straight line in space. Designating the radius of one of the circles by r and the distance of its center from O by d, there is

$$r = \lambda\, d,$$

where λ is a factor of proportionality. Designating the angle of inclination of the straight line with the plane by α,

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on this subject in the *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich: Ein neuer Weg zur Theorie der Kegelschnitte*, Vol. 25, 1880, and *Zur Geschichte und Theorie der elementaren Abbildungsmethoden*, Vol. 27, 1882. See also Fiedler's *Cyclographie*, Leipzig, 1882.

$$tan\ \alpha = \lambda.$$

A plane P in a general position intersects the plane of reference E in a straight line s (Fig 1). For any point A of the plane P and the respective circle, with A' as a center and AA' as a radius, the relation exists that the ratio of the radius AA' and the distance $A'B$ of the center A' from s is constant. Hence, if $< A\ B\ A' = i$,

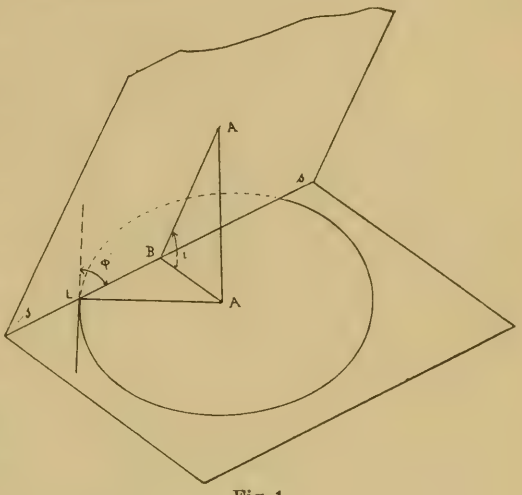


Fig. 1.

$$\frac{AA'}{BA'} = tan\ i,$$

and since $A'L = AA'$, $\frac{A'B}{A'L} = const. = cos\ \phi$, the cosine of the angle which the tangent at L makes with s. Conversely, it can easily be shown that any circle intersecting s at a constant angle ϕ, represents a point of the plane P. Hence the theorem:

The system of all circles intersecting a fixed line in their plane under a constant angle, represents a plane in space.

This theorem is general and applies to all planes in space if imaginary angles of intersection are included. In any real or imaginary case of ϕ, $tan\ i.\ cos\ \phi = 1$. For $\tan i < 1$, i. e., $i < \frac{\pi}{4}$ $cos\ \phi > 1$, so that ϕ becomes imaginary.

3. Complexes and Congruences of Circles in a Plane.

The fact that ordinary space contains a triply infinite number of points, and a plane a triply infinite number of circles, makes it possible to establish a uniform correspondence between the points on one

side of the plane and the circles in this plane. All circles of a plane form a *complex of circles.* An analytic relation

$$F(x, y, z)=0$$

between the Cartesian co-ordinates x, y, z of a point in space defines a surface and limits the points in space to a twofold continuum. The corresponding system of circles, consisting of a doubly infinite number, is called a *congruence of circles.* Assuming the plane of reference as the XY-plane, and writing the equation of the surface in the form

$$z=\psi(x, y),$$

the equation of the congruence of circles is

$$(\xi-x)^2+(\eta-y)^2=[\psi(x, y)]^2,$$

where ξ and η are the running co-ordinates of any point of the congruence. The congruence is of the same order as that of $F(x, y, z)=0$. Thus, in general, about every point of the plane of reference there are n concentric circles of the congruence, n being its order.

As an example, take for $F(x, y, z)=0$ the plane

$$ax+by+cz+d=0.$$

The equation of the congruence becomes

$$(\xi-x)^2+(\gamma-y)^2=\frac{(ax+by+d)^2}{c^2}$$

The equation of s is

$$ax+by+d=0,$$

and the distance of the center (x, y) from s is, Fig. 1.

$$A'B=\frac{ax+by+d}{\sqrt{a^2+b^2}}.$$

The radius $r=A'L=\frac{ax+by+d}{c}$, hence

$$\cos\phi=\frac{c}{\sqrt{a^2+b^2}},$$

which shows that all circles of the congruence intersect the line s under a constant angle, which is real or imaginary according, as

$$a^2 + b^2 \gtrless c^2, \quad \text{respectively.}$$

(See geometrical proof under 2).

4. Linear Congruence of Circles.

Let $X^2 + Y^2 = R^2$ be the equation of a fixed circle, and

$$(X-x)^2 + (Y-y)^2 = r^2$$

the equation of any circle orthogonal to the first. The condition for orthogonality is

$$x^2 + y^2 = R^2 + r^2.$$

In the cyclographic interpretation of space we have to put $r = z$, so that

$$x^2 + y^2 - z^2 = R^2.$$

This is the surface, $F(x, y, z) = 0$, and represents an orthogonal hyperboloid of rotation of one nappe, having the z-axis as an axis of rotation.

A system of circles representing such a surface is called a *linear congruence of circles*. If we now introduce another condition,

$$G(x, y, z) = 0,$$

the circles of the congruence limited by this condition will be singly infinite in number. Their centers are situated on a certain curve (projection of intersection of $x^2 + y^2 - R^2 = 0$, and $G(x, y, z) = 0$), and they may envelope a curve. If $G = 0$ is a plane, the locus of the centers is a *conic* and the envelope a *bicircular quartic.*

In tetracyclic co-ordinates[1] the equation of the linear congruence may be written

$$x_1 m + x_2 m_2 + x_3 m_3 + x_4 m_4 = 0,$$

[1] See M. Böcher: Uber die Reihenentwickelungen der Potentialtheorie, Chapts. 1 and 2.
G. Darboux: Théorie Générale des Surfaces, Vol. I, Chap. VI.

where the m_k's satisfy the linear equation

$$a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 = 0.$$

All circles of the congruence are orthogonal to the fixed circle with the equation

$$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = 0.$$

Thus in corroboration of the previous result we may state the theorem:

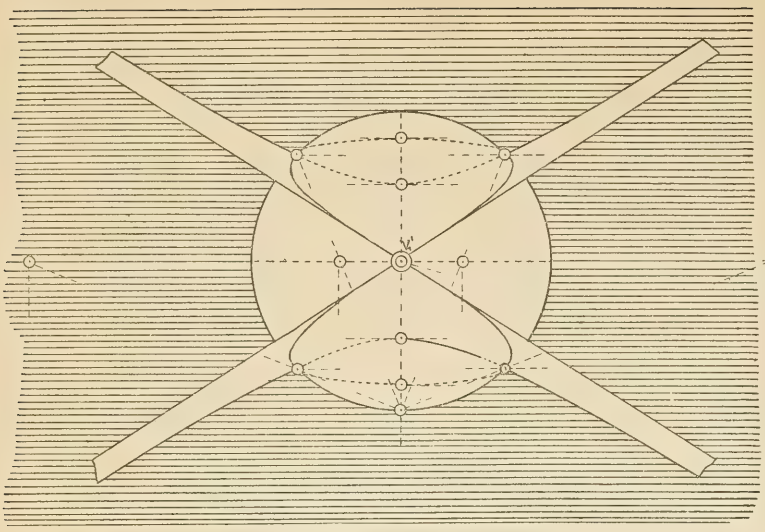
A linear congruence of circles in a plane consists of all circles orthogonal to a fixed circle in the same plane.

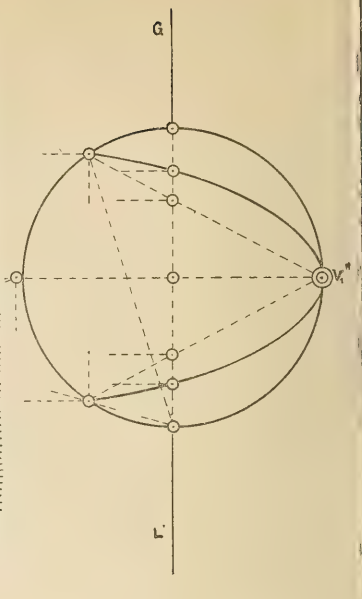
The properties of this congruence may be studied in a simple manner by considering with M. Bôcher, loc. cit., the sphere

$$\sum_{1}^{4} {}_{k}^{2} x = 0$$

and its stereographic projection. Any cone of the second order with the vertex V_1 intersects the sphere in a sphero-quartic C_4. It is well known that there are three other cones of the second order with the vertices V_2, V_3, V_4 which pass through the same sphero-quartic. The tetrahedron V_1, V_2, V_3, V_4 is the self-polar tetrahedron of the pencil of quadrics passing through C_4. Now the polar-reciprocal surfaces, with respect to the sphere, of the 4 cones are four conics K_k in the planes of the polar tetrahedron. The planes of this tetrahedron intersect the sphere in four mutually orthogonal circles. The tangent planes of each cone intersect the sphere in circles which touch C_4 in every case in two points, and whose poles are situated on the reciprocal surface of the cone—in this case on a conic. Making a stereographic projection of this configuration with respect to the original sphere, the theorem may be deduced:

The circles of a linear congruence whose centers are situated on a conic, envelope a bi-circular curve of the fourth order. The same curve is also produced by three other systems of circles belonging to three different congruences, and having their centers on three different conics, respectively. The fundamental circles of these four congruences are mutually orthogonal.

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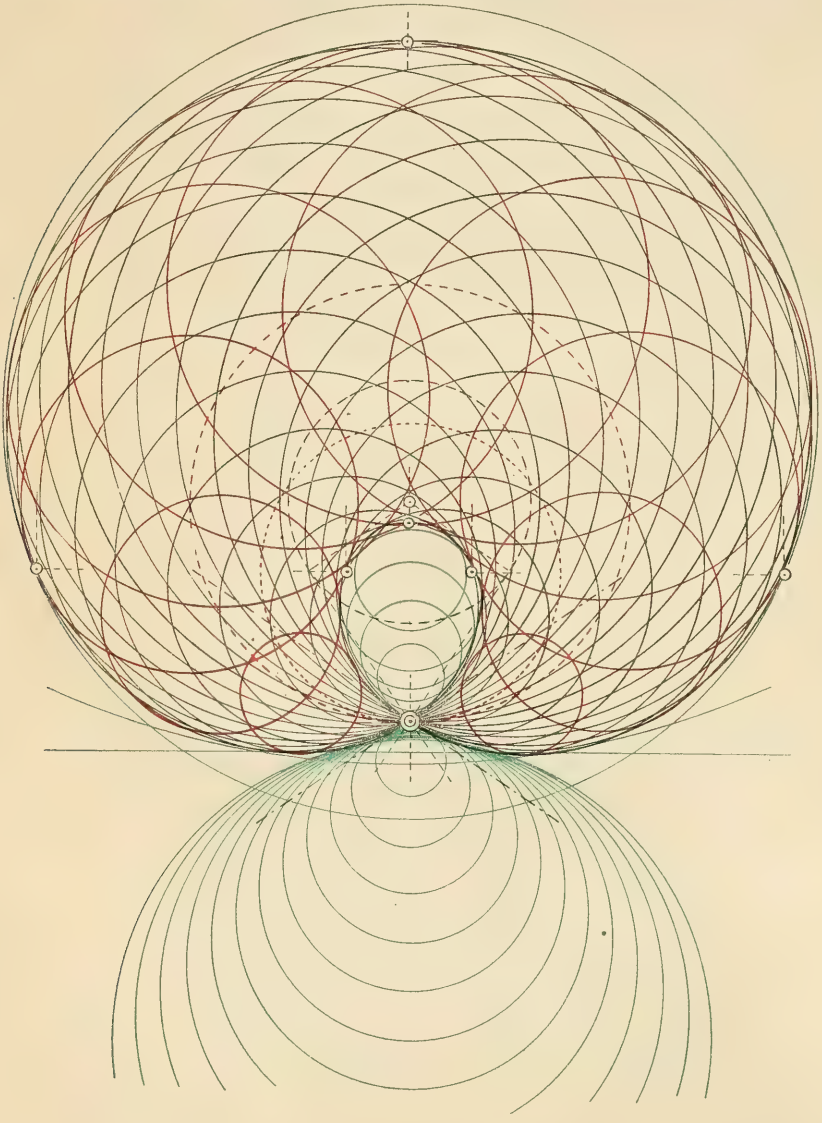


The equation of the bicircular quartic is

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 = 0,$$

and has the four orthogonal circles as co-ordinate circles.

We shall illustrate these results by an example. Assume as one of the cones of intersection an hyperbolic cone having its vertex V_1 on the sphere. In the adjoining plate a symmetrical arrangement has been effected. The horizontal plane of projection has been assumed through the center of the sphere, and the base of the cone (V_1) appears as a hyperbola in the hotizontal plane. The vertical projection of the C_4 is an arc of an ellipse. The same is true of the profile-projections (not shown in the figure). The tangents at the double-point of the horizontal-projection of C_4 are the asymptotes of the base-hyperbola.

The four cones (V_1), (V_2), (V_3), (V_4), consist, therefore, of the original cone, which is double-counting (V_1), (V_2), and two elliptical cylinders (V_3) and (V_4), so that only three systems of circles are obtained. The stereographic projection was determined by taking the profile-plane through the center of the sphere as a plane of projection. In the plate the three systems of circles are distinguished by black, red and green lines, respectively.

The stereographic projection of this plate admits of an interesting interpretation by the cyclographic method. Consider any of the systems of circles—for instance, the red circles. As these are all orthogonal to a fixed circle (red dot), they represent points of an orthogonal hyperboloid of rotation of one nappe. As their centers are on a conic (red dash), they also represent points on a cylinder. This system of circles consequently represents a twisted curve of the fourth order C_4', the intersection of the cylinder and the hyperboloid. It follows, therefore, from my previous paper, that *the bi-circular quartic in the plane of projection is the trace of the surface of singularities of the cyclographic congruence defined by the twisted curve* C_4', *in space.*

5. A General Theorem.

We propose now to find the envelope of all circles which cut a given circle orthogonally and whose centers are situated on a curve of the n^{th} order in the plane of the circle. Let C_n be the given curve and C the given circle. Find the stereographic projection Q of C on the sphere $S\left(\sum_1^4 x_k^2 = 0\right)$, and determine the pole V of the plane E through Q. Project the curve C_n stereographically upon E, thus producing a curve C_n' of the same degree. The polar surface of C_n' with regard to S is a cone of class n having V as a vertex. Every tangent plane of this cone cuts S' in a circle K which is orthogonal to Q and whose pole is a point of C_n'. The stereographic projection of K is a circle which cuts C orthogonally and whose center lies on C_n. The system of circles K on S envelope a curve of order $2m$, if m is the order of the cone of class n. Hence the theorem:

The envelope of all circles whose centers lie on a curve of the n^{th} order and which cut a given circle orthogonally, is a curve of order $2m$ if m is the class of the curve of order n[1].

If the given curve has δ double points and k cusps,

$$m = n\,(n-1) - 2\delta - 3k.$$

The order of the envelope is, therefore,

$$2\,m = 2\,[n(n-1) - 2\delta - 3k].$$

Taking a bi-circular quartic with a real double-point,

$$2m = 2[4.3 - 2.3] = 12.$$

In general the trichodals (lemniscates) give an envelope of order 12, and also the Cartesians. The limaçon gives rise to a curve of order 8, and the cardioide of order 6.

The cyclographic interpretation of the above theorem may be stated in the form:

[1] This theorem has also been proved analytically by Dr. Virgil Snyder, of Cornell University, who has published a number of valuable papers on the geometry of the circle. See *Bulletin of the Am. Math. Soc.*, Vol. VI., pp. 319-322.

The trace of the singular surface of the cyclographic congruence based upon the curve of intersection of an orthogonal hyperboloid of rotation of one nappe and a right cylinder of the n^{th} order or m^{th} class is a curve of the $2m^{th}$ order.

6. General Representation of Curves.

A curve in space may always be considered as the intersection of two surfaces

$$F(x, y, z) = o \text{ and } G(x, y, z) = o.$$

The system of circles in the plane of reference, representing the curve, is therefore common to two linear congruences and may have an envelope. Representing the curve by

$$x = f(t),\ y = g(t),\ z = h(t),$$

where t is a parameter, the equation of the system of circles becomes

$$[x - f(t)]^2 + [y - g(t)]^2 = [h(t)]^2.$$

The envelope is obtained by eliminating t from this equation and

$$[x - f(t)]\, f'(t) + [y - g(t)]\, g'(t) = h(t)\, h^1(t).$$

In general this elimination is difficult, so that other methods must be applied.

7. Representation of Conics.

The circles representing a plane curve, all intersect a certain straight line of the plane of reference under a constant angle. Thus, the system of circles intersecting a given straight line under a constant angle (real or imaginary) and having their centers on a conic, represent a conic in space. The determination of the envelope of the system of circles representing a conic in space requires the consideration of the congruence of rays which pass through two given conics in space. Through every point in space (not on one of the conics) four rays of the congruence may be passed and in every plane lie six rays. Hence, the congruence is of the 4th order and 6th class. The

two nappes of the focal surface are formed by the two given conics, and the developable surfaces consist of the cones having their centers on these conics and passing through these. The singular surface of the congruence, or the envelope of these cones, is itself developable and is generally of the 4th class and 8th order. If the plane of the conic is inclined 45°, the envelope of the circles is of the 4th class and the 6th order.

8. Parallel Curves.

In case that the curve is parallel to the plane of reference, we may write the equation of the curve in the form

$$\xi = f(t),\ \eta = g(t),$$

so that the equation of the representative system of circles becomes

$$[x - f(t)]^2 + [y - g(t)]^2 = r^2 \ (const.)$$

By differentiation

$$[x - f(t)]f'(t) + [y - g(t)]\,g'(t) = o.$$

From these equations the co-ordinates x and y of a point of the envelope are found.

$$x = f(t) \pm \frac{r\,g'(t)}{\sqrt{f'(t)^2 + g'(t)^2}},$$

$$y = g(t) \pm \frac{r\,f'(t)}{\sqrt{f'(t)^2 + g'(t)^2}},$$

These are, however, also the values of the co-ordinates of a point Q on the normal through P at a distance $\pm r$ from P. Hence the theorem:

The envelope of a system of equal circles having their centers on a plane curve and being situated in the plane of the curve is identical with the curve described by the center of an equal circle which rolls upon the given curve.

In case of a conic the envelope is a curve of the 8th order and 4th class. The normals to the conic are the projections of the generatrices of the corresponding developable singular surface of the congruence. The edge of regression of this surface is orthographically projected as the evolute of the conic. All circles tangent to a conic represent a developable surface of the 8th order and 4th class.

Numerous other applications might be added which would show the usefulness of the method of cyclography for the solution of certain problems of historic interest. Fiedler, for instance, has given a remarkably simple solution of Apollonius' problem by this method. The propositions of this paper concerning orthogonality may also be generalized for isogonality. In a future article the writer intends to continue this subject.

May 21, 1901.

TWO NOTES ON THE ELLIPSE

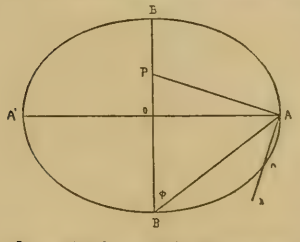
EDWARD L. BROWN

I.

Salmon, in his Conic Sections, Art. 179, draws attention to the fact that we can draw a pair of conjugate diameters intersecting at any other angle, making use of the truth that diameters parallel to any pair of supplemental chords are conjugate. For, if we describe on any diameter a segment of a circle, containing the given angle, and join the points where it meets the curve to the extremeties of the assumed diameters, we obtain a pair of supplemental chords inclined at the given angle, the diameters parallel to which will be conjugate to each other.

From the above, readers are almost always of the opinion that the circle will in every case intersect the ellipse in points other than the extremeties of the assumed diameter, and that the construction is possible for any angle whatever. This, however, as is well known, is not true. The following is a discussion of the construction:

Let $A\ A^1$ be the assumed diameter. At A construct an angle $O\ A\ X$, which is to be equal to the angle between the conjugate diameters. Join A and B. Draw $P\ A$ perpendicular to $A\ X$. $A\ P$ will be the radius of the circle to be constructed, a segment of which will contain the given angle.



If the angle $P\ A\ B$ is greater than the angle $P\ B\ A$, then will the line $P\ B$ be longer than the line $P\ A$. In this case the circle will intersect the ellipse in the points A and A^1 only. For, if the circle cut the ellipse in points other than A and A^1, they would be symmetrically situated with respect to B. This is impos-

sible, since the radius is less than $P\ B$, and no part of the circle can lie below B. Nor can the circle intersect the upper part of the ellipse; for, if it did, the points of intersection would be symmetrical with respect to the point B. This would cause an arc of the circle to pass between P and B^1, which is impossible, since $P\ A > A\ O > O\ B^1 > P\ B^1$. Therefore, when the angle $P\ A\ B$ is greater than the angle $P\ B\ A$, the circle intersects the ellipse in the points A and A^1, and in no other points. In this case the construction is impossible.

If the angle $P\ A\ B$ equals the angle $P\ B\ A$, then will $P\ A = P\ B$, and the circle will be tangent to the ellipse at B. In this case the circle intersects the ellipse in four points — A, A^1, and two coincident points at B. Since it intersects the ellipse in these four points, it cannot intersect the ellipse above the X-axis, as is also evident since the angle $O\ B\ A$ is greater than the angle $B\ A\ O$. In this case the construction is possible.

If the angle $P\ A\ B$ be less than the angle $A\ B\ P$, then will $P\ B$ be shorter than $P\ A$. In this case the circle must intersect the quadrants $A\ B$ and $A\ B^1$ in points symmetrical with respect to B. It is evident that this will be the case; for the line $A\ X$, perpendicular to $A\ P$, will always intersect the ellipse. And since the circle described with $A\ P$ as a radius must lie in the angle $P\ A\ X$, it must intersect the ellipse between K and B, and in a point in the quadrant $B\ A^1$. In this case, also, the construction is possible.

We have seen that the greatest value the angle $P\ A\ B$ can have is the value of the fixed angle $A\ B\ P$. From this it follows that the angle $A\ P\ B$ is a minimum when $A\ P = B\ P$. This is the minimum value of the constructed angle $O\ A\ X$, which is equal to the angle $A\ P\ B$, leaving their sides perpendicular to each other.

Now, let us find the value of this constructed angle when a minimum.

$$O\ A\ X = 180 - 2\ \phi$$

$$\therefore \text{Sin}\ O\ A\ X = \text{Sin}\ 2\ \varphi = 2\ \text{Sin}\ \phi\ \cos\ \phi = \frac{2\ ab}{a^2 + b^2}$$

This same result is easily obtained from $a^1\ b^1\ \sin\ \theta = ab$, where

a^1, b^1, are semi-conjugate axes, and θ the angle included by them. Sin θ is a minimum when $a^1\ b^1$ is a maximum. This will be the case when $a^1 = b^1$; for

$$(a^1 - b^1)^2 + 2\ a^1\ b^1 = a^{\dot{1}} + b^{\dot{1}} = \text{constant.}$$

Now $2\ a^1\ b^1$ is a maximum when

$$(a^1 - b^1)^2 = 0.$$

That is when $a^1 = b^1$.

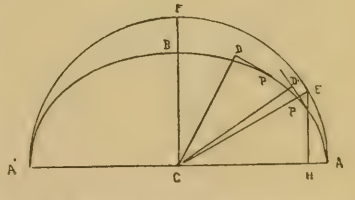
$$\therefore \quad a^2 + b^2 = a^{\dot{1}} + b^{\dot{1}} = 2a^{\dot{1}}.$$

$$\therefore \quad a^{1\,2} = \frac{a^2 + b^2}{2}.$$

Then we have

$$\text{Sin } \theta = \frac{ab}{a^1\ b^1} = \frac{ab}{a^{1\,2}} = \frac{2ab}{a^2 + b^2}$$

II.



On $A\ A^1$ as a diameter, construct the semi-circumference $A\ F\ A^1$, and the semi-ellipse $A\ B\ A^1$. Construct the angle $B\ C\ E$ = angle $A\ C\ D$. Draw $E\ H$ perpendicular to $A\ A^1$, intersecting ellipse in P^1. Draw $C\ D$ and $C\ D^1$ perpendicular to the tangents at P and P^1, respectively.

Let ϕ = eccentric angle of P.
ϕ^1 = eccentric angle of P^1.
a = angle $D\ C\ A$.

By Fagnani's theorem we have

$$B\ P - A\ P^1 = P\ D$$

Comparing the two forms for the equation of the tangent, viz.:

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

$$x\cos\alpha+y\sin\alpha=p$$

we get

$$\frac{\cos\phi}{a}=\frac{\cos\alpha}{p},\quad \frac{\sin\phi}{b}=\frac{\sin\alpha}{p} \tag{1}$$

Observing that $\sin\alpha=\cos\phi^1$, equations (1) show that

$$\frac{\cos\phi}{a}=\frac{\sin\phi^1}{p},\quad \frac{\sin\phi}{b}=\frac{\cos\phi^1}{p}$$

By eliminating p, we get

$\tan\phi.\ \tan\phi^1=\frac{b}{a}$, a relation connecting the eccentric angles of P and P^1.

This result can probably be made useful, and so far as I know, is new.

NOTES ON EARLY GREEK COSMOGONICAL SPECULATIONS

GEORGE NORLIN

It is my purpose to point out briefly certain features that run through the early attempts of the Greeks to picture to themselves the creation of the world and the origin of things.

In view of the prevalent notion that there is little or nothing in common between the fantastical dreams of the poets on this subject and the more sober speculations of the early philosophers, it may not be without interest to note that there are common features that characterize the conception of the poet and philosopher alike, and serve, in some degree at least, to bridge over the apparent gulf between them.

1. The *method* of the early philosophers is not *essentially* different from the poet's in trying to solve the problem of the origin of things. Thinkers like Anaximander and Heraclitus are virtually poets who try to *imagine* how this world might have sprung into being. They abandoned, it is true, the cruder fancies of the popular cosmogonies handed down by the poets, and assumed a first principle or fundamental substance from which the things of this world sprang and of which they are but transitory forms. However, this first principle once assumed, they deduced the world of things from that with a spirit and a method not essentially different from the poet's. There was as yet no patient seeking after facts and building up the structure of knowledge stone by stone.

Rather, there followed the first awakening of the philosophic movement an intoxication of mind which recognized no limitations to human knowledge, but with the poet's boldness of imagination sought at once to grasp the great things of the world.

In a word, then, the speculations of the early philosophers are not radically differentiated from the fancies of the poets by the logical consistency and sober discrimination which come only from a long period of training in rigid scientific method. The early philosopher will be best understood if we think of him as gifted with the eager desire to *know*, together with the poet's fancy and that adventurous imagination which, to use the words of the fine tribute of Lucretius to his master:

* * * * * * Extra
Processit longe flammantia moenia mundi
Atque omne immensum peragravit mente animoque,
Unde refert nobis victor quid possit oriri,
Quid nequeat, finita potestas denique cuique
Quanam sit ratione atque alte terminus hærens.[1]

2. The poet and philosopher alike, in thinking about the origin of things, assume the existence of matter at the first. There is no creation out of nothing. It is surprising to find in so well-known a writer as John Fiske a statement to the contrary: "The Ancients freely admitted that matter might be created and destroyed.[2]" If he means in these words to include the Greeks, he could not be further from the truth. Burnet, our best authority on early Greek philosophy, says: "The great principle which underlies the speculations of the early cosmologists, though it is first laid down by Parmenides, is that 'Nothing comes into being out of nothing, and nothing passes away into nothing.[3]' "

This statement of Burnet needs some qualification. Anaximander, for instance, had the first part of the dictum but not the last. He assumed that nothing could come out of nothing, but seems to have held that matter can be destroyed. At any rate, that seems to be the implication of the statement that Anaximander assumed his primary substance or first principle to be infinite, in order that the process of

[1] Lucret. De Rerum Natura I:73-77.
[2] Cosmic Philosophy, I. p. 280.
[3] Early Greek Philosophy, p. 9.

coming into being might never cease.[1] This can hardly mean anything else than that the supply of matter would in time give out unless it were infinite in quantity. Such an assumption, as Aristotle points out[2], is entirely unnecessary if the indestructibility of matter be assumed.

The statement of Burnet is, however, generally true for early philosophy. The doctrine that nothing can come into existence out of nothing and nothing can pass away into nothing, seems, with the exception above pointed out, to have been a tacit assumption in early philosophy, until it finds explicit statement in Parmenides, after which it becomes recognized as a fundamental law.

For the first part of the doctrine, however, that nothing can be created out of nothing, we need not make an exception even of Anaximander. He himself would subscribe to the words of Lucretius:

> Principium cuius hinc nobis exordia sumet,
> Nullam rem e nilo gigni divinitus umquam.[3]

The poet, antedating the philosopher in his mythical account of the origin of the world, does not, of course, state this doctrine, nor is he conscious of it as a principle, but he nevertheless unconsciously assumes it. For he starts, in his picture of the creation of the world, not with a god who creates, but with matter out of which is created, though, as we shall see, he rather vaguely and crudely personifies his primal matter into a god.[4] To take only a single instance: Hesiod, in his Theogony, starts with Chaos, a great gulf filled with dark

[1] Plac. I 3, 3. Diels, Doxagr. 277. *Ἀναξίμανδρος δὲ ὁ Μιλήσιός φησι τῶν ὄντων τήν ἀρχὴν εἶναι τὸ ἄπειρον. * * * λέγει γοῦν διότι ἀπέραντόν ἐστιν, ἵνα μηδὲν ἐλλείπῃ ἡ γένεσίς ἡ ὑφισταμένη.*

[2] Phys. III 8. 208ª 8. *οὔτε γὰρ ἵνα ἡ γένεσις μὴ ἐπιλείπῃ, ἀναγκαῖον ἐνεργείᾳ ἄπειρον εἶναι σῶμα αἰσθητόν. ἐνδέχεται γὰρ τὴν θατέρου φθορὰν θατέρου εἶναι γένεσιν, πεπερασμένου ὄντος τοῦ πάντος.*

[3] De Rerum Natura I. 149–150.

[4] Preller, Griechische Mythologie, 4th Ed., p. 30. Zwei Grundgedanken gehen durch das Ganze. Der erste ist dass die Welt nicht auf einmal d. h. durch Schöpfung entstanden, sondern aus dunklen und elementaren Aufängen durch organische Entwick-

mist, or matter vaguely defined. Out of this we have a series of births until we come to Heaven and Earth and the race of the gods.

3. Poet and philosopher alike vaguely identify matter and force. In the case of the poet, this, perhaps, goes without saying. It is to be expected that writing before the time of philosophic reflection began, he should have explained the creation of the world in terms of the gods. The Chaos of Hesiod is not merely space filled with matter undefined, but is a god who brings to birth out of himself. The great parents, Heaven and Earth, moreover, are themselves gods by more than mere personification. The poet's interpretation of the world we may call animistic. The great forces that are at work in nature, the apparent living activity of the world, are most easily explained in his mind as the manifestations of divine power.

This animism of the poet finds its analogue in the hylozoism of the philosopher. Thales, for instance, seems to have held, to translate quite literally the words of Aristotle, that "All things are filled with gods." The mysterious power of the magnet he explained as due to the presence of a spirit in it.[1]

The philosopher's first principle or fundamental substance is itself instinct with life, and develops the world of individual things out of itself by its own inherent power. Anaximander speaks of his *ἄπειρον* or indeterminate matter, as a Divinity which "embraces all things and pilots all things."[2]

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elungen bis zu dieser letzten Gestalt des schönen vollendeten Kosmos gediehen ist, deren endliche Spitze und Vollendung eben Zeus und die von ihm regierte Welt der Götter und der Natur ist.

[1] Aristotle, de Anima I 5. 411ª 7. *καὶ ἐν τῷ ὅλῳ δέ τινες αὐτὴν (τὴν ψυχὴν) μεμῖχθαί φασιν, ὅθεν ἴσως καὶ Θαλῆς ᾠήθη πάντα πλήρη θεῶν εἶναι.* cf. de Anima I 2. 405ª 19. *ἔοικε δὲ καὶ Θαλῆς ἐξ ὧν ἀπομνημονεύουσι κινητικόν τι τὴν ψυχὴν ὑπολαβεῖν, εἴπερ τὸν λίθον ἔφη ψυχὴν ἔχειν, ὅτι τόν σίδηρον κινεῖ.*

[2] Aristotle, Phys. III 4. 203ᵇ 7. *αὕτη τῶν ἄλλων εἶναι δοκεῖ καὶ περιέχειν ἅπαντα καὶ πάντα κυβερνᾶν, * * * καὶ τοῦτ' εἶναι τὸ θεῖον.* cf. Simpl. Phys. 465, 13D.

So, also, Anaximines called Air, the first principle or fundamental substance with which he starts, a god.[1]

Other illustrations might be given, but these will perhaps suffice.

In Empedocles, we have the first attempt to conceive of force outside of matter. His two forces, Love and Hate, φιλότης καὶ νεῖκος, work on the four elements, Earth, Air, Fire and Water; but these forces, as Burnet shows, are themselves more or less material. A more successful attempt at conceiving immaterial force is the νοῦς, or intelligence, of Anaxagoras; but here again there is confusion of force with matter.

The atoms of Democritus are explicitly stated to be devoid of sensation, volition or intelligence, and yet they possess the inherent tendency to fall—a tendency he does not explain, though he is vaguely groping for the idea of gravitation.

5. The poetic and philosophic cosmogonies are both characterized by the conception of a dual agency at the beginning of the cosmic process. The first parents are for the poet Darkness and Light, variously named, sometimes called Night and Day; sometimes Chaos and Æther; and again Night and Phanes. These agents are always personified: Darkness, the passive agent; Light, the active. They are, at the same time, vaguely conceived as material things, Darkness being usually conceived as a *murky, indistinguishable mist* in which the vivifying ethereal Light is latent. This dualism is found in a less fantastic form in the philosophic conception also, as I shall try to show, after giving illustrations of it from the poet's version of the world story.

Milton, in "Paradise Lost," expresses the Greek mythical conception when he starts with a dark

[1] Cicero Deor. Nat. I 26. Anaximines aera deum statuit eumque gigni esseque immensum et infinitum et semper in motu. cf. Augustine Civ. D. VIII 2. Omnes rerum causas aeri infinito dedit nec deos negavit aut tacuit, non tamen ab ipsis aerem factum sed ipsos ex aere ortos credidit.

"Illimitable ocean without bound
Without dimension, where length and breadth and height
And time and place are lost, where eldest night
And chaos, ancestors of nature, hold
Eternal anarchy."

The first step in the creation of the world is taken when

"At last the sacred influence of light appears,
And from the walls of heaven,
Shoots far into the bosom of dim night,
A glimmering dawn."

The Chaos with which Hesiod starts[1] is virtually the "Illimitable Ocean" of Milton "Where eldest night and chaos hold eternal anarchy."[2]

From this Chaos are born Erebos and Nux, both powers of darkness; then are born Æther and Hemera, Αἰθὴρ καὶ Ἡμέρα, male and female principles of light.

In the Orphic cosmogony according to Eudemus, Night is the starting point,[3] but unfortunately we have no account of anything further in this version.

In the Orphic cosmogony known as the Rhapsodic cosmogony, we start practically with Chaos and Æther. The Orphic Chaos is, like the Hesiodic, the great yawning gulf filled with dark misty

[1] Theog. 116 ff.

[2] Scholiast to Apoll. Rhod. I. 498. καί Ζήνων καὶ τὸ παρ' Ἡσιόδῳ χάος ὕδωρ εἶναί φησιν, οὗ συνιζάνοντος, ἰλὺν γίνεσθαι, ἧς πηγνυμένης γῇ στερεμνιοῦται. cf. Theog. 814. χάεος ζοφεροῖο. cf. for much the same conception, Genesis I, 2. "And darkness was upon the face of the deep." cf. Vedic Hymn R. V. X, 129, containing the germ of all Vedic cosmogonical theories. "Darkness existed; originally enveloped in darkness the universe was indistinguishable water." Trans. John Muir, original Sanscrit Texts, IV, p. 4.

[3] Abel, Orphica, frag. 30.

matter.[1] We have, then at the first, in this cosmogony, formless darkness and mist existing together with Æther, the pure ethereal light. These bring forth the world egg[2] from which springs Phanes, the bright, shining one, giver of light and of life.[3] Phanes then brings forth Night, and with her in the dark misty cave, *σπέος ἠεροειδές*, brings forth the race of the gods.[4] This is evidently a repetition of the old story of Light and Darkness. Phanes and Night are practically the same as Æther and Chaos.

The story of Darkness and Light at the beginning is told in the cosmogony of Acusilaus. Here we have in the main a repetition of Hesiod.

Chaos is the first principle, and from Chaos are born a pair, Erebos and Nux, and from them spring Æther, and Love, and Metis.[5]

In the cosmogony of Epimenides, the first principles are Air and Night. His Air, *ἀήρ*, the misty atmosphere, as distinguished from the bright transparent *αἰθήρ*, is equivalent to the *σκοτοέσσα ὀμίχλη* of the Orphic doctrine. From these two spring Tartarus and two beings undefined who produce the world egg from which springs the race of the gods.[6]

In the very fanciful cosmogony found in Aristophanes' Birds, evidently a travesty on early mythical cosmogonies, we have at the first, Chaos and Night, Erebos and Tartarus. Here, under four names, we surely have darkness enough to begin with. Night begets a wind egg from which springs Eros, who, like the Phanes of the Orphic account, is a deity of light and of life. By mingling things together, he produces gods and men.[7]

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[1] It is called *μέγα χάσμα πελώριον* and *σκοτόεσσα ὀμίχλη*. Abel, Orphica, frag. 52.
[2] Abel, Orphica, frag. 53.
[3] Abel, Orphica, frags. 56, 58, 59, 61.
[4] Abel, Orphica, frags. 72, 73, 89.
[5] Damascius, c. 124; Lukas, Kosmogonien der Alten Völker, p. 162.
[6] Damascius, c. 124; Lukas, Kosmogonien der Alten Völker, p. 176.
[7] Aristophanes' Birds, 693 ff.

So much for the mythical notions embodied for the most part in the poets. The philosopher's view is naturally more sane and sober, less fantastical and crude. But the old dualism of Darkness and Light persists in a more abstract form. Darkness and light now become *qualities* of the two opposite *elements*, which together work out the "scheme of the world." With the one element are conventionally associated the qualities cold, dark, dense, moist, heavy; with the other, the qualities hot, bright, rare, dry, light. These opposites with which the philosophers start are virtually the first parents of the poets, Darkness and Light or Night and Day.

The general notion of the philosophers was that the cold, dark, dense, moist, heavy element swung to the center and formed the earth with its waters and atmosphere, while the warm, dry, bright, rare, light element flew to the outermost regions of space and formed the bright transparent æther and the fiery bodies of heaven.

In Anaximander we have the opposites—hot and cold. The cold element forms the earth and its atmosphere. The hot, at first a hollow sphere of fire, surrounds the earth and its atmosphere as "the bark surrounds a tree." This outer crust of fire is then broken up and forms the sun, moon and stars.[1]

Parmenides furnishes a still better illustration. He starts with two opposite elements, which he variously names ethereal, flaming fire, light and fleet,[2] and dark night, a dense and heavy body;[3]

[1] Diels' Dox. 579, 13, quoted by Ritter and Preller, Historia Philosophiae Græcae, p. 17. *Φησὶ δὲ τὸ ἐκ τοῦ ἀιδίου γόνιμον θερμοῦ τε καὶ ψυχροῦ κατὰ τὴν γένεσιν τοῦδε τοῦ κόσμου ἀποκριθῆναι καί τινα ἐκ τούτου φλογὸς σφαῖραν περιφυῆναι τῷ περὶ τὴν γῆν ἀέρι ὡς τῷ δένδρῳ φλοιὸν. ἥστινος ἀπορραγείσης καὶ εἴς τινας ἀποκλεισθείσης κύκλους ὑποστῆναι τὸν ἥλιον καὶ τὴν σελήνην καὶ τοὺς ἀστέρας.*

[2] *φλογὸς αἰθέριον πῦρ * * μέγ' ἐλαφρόν*, l 116.

[3] *νύκτ' ἀδαῆ, πυκινὸν δέμας ἐμβριθές τε*, l 119.

or simply light and night;[1] or fire and night;[2] or again fire and earth.[3]

Fire is the active, Night the passive element, and from a union of these opposites spring all things.[4]

Anaxagoras distinguishes as his two opposites, Air and Æther, the one, dark, moist, dense and cold; the other, bright, dry, subtile and warm. The former segregates in the center of the world where it forms the earth; the latter, in the outer regions of space.[5]

Space prevents my giving further illustrations of this dualism in other philosophers. I shall have to content myself with citing, in briefest form, the principal references.[6]

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[1] 1 123.

[2] 1 126.

[3] Simplicïus, Phys. 146, 26 D. R. and P., p. 94. *καὶ γὰρ Παρμενίδης θερμὸν καὶ ψυχρὸν ἀρχὰς ποιεῖ, ταῦτα δὲ προσαγορεύει πῦρ καὶ γῆν.*

[4] Theophrastus in Diels, Dox. 482. R. and P. p. 95. *κατὰ δόξαν δὲ τῶν πολλῶν εἰς τὸ γένεσιν ἀποδοῦναι τῶν φαινομένων, δύο ποιῶν τὰς ἀρχάς, πῦρ καὶ γῆν, τὸ μὲν ὡς ὕλην, τὸδ' ὡς αἴτιον καὶ ποιοῦν.* cf. Cicero, Acad. II, 118. Parm. ignem qui moveat, terram quæ ab eo formetur. Plut. adv. Col. 13, 6. R. and P. p. 95. *ὅς γε καὶ διάκοσμον πεποίηται, καὶ στοιχειᾶ μιγνύς, τὸ λαμπρὸν καὶ σκοτεινόν ἐκ τούτων τὰ φαινόμενα πάντα καὶ διὰ τούτων ἀποτελεῖ.*

[5] Theophrastus in Diels, Dox. 516, 5. R. and P. p. 119. *τὸ μὲν μανὸν καὶ λεπτὸν θερμόν, τὸ δὲ πυκνὸν καὶ παχὺ ψυχρόν, ὥσπερ Ἀναξαγόρας διαιρεῖ τὸν ἀέρα καὶ τὸν αἰθέρα.* Diels, Dox. 561, 27. R. and P. p. 119. *τὸ μὲν οὖν πυκνὸν καὶ ὑγρὸν καὶ τὸ σκοτεινὸν καὶ ψυχρὸν καὶ πάντα τὰ βαρέα συνελθεῖν ἐπὶ τὸ μέσον, ἐξ ὧν παγέντων τὴν γῆν ὑποστῆναι· τὰ δ' ἀντικείμενα τούτοις τὸ θερμὸν καὶ τὸ λαμπρὸν καὶ τὸ ξηρὸν καὶ τὸ κοῦφον εἰς τὸ πρόσω τοῦ αἰθέρος ὁρμῆσαι.*

[6] On Anaximines, Diels, Dox. 560,13. *ὥστε τὰ κυριώτατα γῆς γενέσεως ἐναντία εἶναι θερμόν τε καὶ ψυχρόν.*

On Zeno, Diog. IX, 29. R. and P. p. 105. *γεγενῆσθαι δὲ τὴν τῶν πάντων φύσιν ἐκ θερμοῦ καὶ ψυχροῦ καὶ ξηροῦ καὶ ὑγροῦ λαμβανόντων εἰς ἀλλήλα τὴν μεταβολήν.*

On Diogenes of Apollonia, Diels, Dox. 583. R. and P. p. 174. *κοσμοποιεῖ δὲ οὕτως· ὅτι τοῦ παντὸς κινουμένου καὶ ᾗ μὲν ἀραιοῦ ᾗ δὲ πυκνοῦ γενομένου, ὅπου συνεκύρησε τὸ πυκνόν, συστροφὴν*

ποιῆσαι καὶ οὕτω τὰ λοιπὰ κατὰ τὸν αὐτὸν λόγον, τὰ κουφότατα τὴν ἄνω τάξιν λαβόντα τον ἥλιον ἀποτελέσαι.

On Archelaus, Diels, Dox. 563. R. and P. p. 176. εἶναι δὲ (δύο) ἀρχὰς τῆς κινήσεως (ἃς) ἀποκρίνεσθαι ἀπ' ἀλλήλων τὸ θερμὸν καὶ τὸ ψυχρὸν, καὶ τὸ μὲν θερμὸν κινεῖσθαι, τὸ δὲ ψυχρὸν ἠρεμεῖν.

Diels, Dox. 280. R. and P. p. 176. τούτων δὲ τὸ μὲν εἶναι πῦρ τὸ δὲ ὕδωρ.

On Hippo, Diels, Dox. 566. R. and P. p. 178. Ἵππων δε ὁ Ῥηγῖνος ἀρχὰς ἔφη ψυχρὸν τὸ ὕδωρ καὶ θερμὸν τὸ πῦρ.

PLAY

ARTHUR ALLIN

The phenomena of play are closely connected with the law of increase of plastic endowment,[1] inasmuch as the latter implies a long period of preparation for the social activities of adult society. It is these adult social activities which set the goal and prescribe the ideals to be attained during the period of youth. Play, as an activity of youth, is an initiation into society. Habits are formed which later may be switched off and attached to other objects and aims needful in the social life of the adult. The house being built may for a time be occupied by the masons and carpenters, but these strange guest-builders soon give place to those tenants for whom the house was originally intended. Throughout all play runs the great principle of vicarious stimuli. The colored plaything of the child, the coveted banner of the college rush, the prize of the physical contest, the victorious score of the billiard player call forth certain activities and habits of reaction which later may be attached to the so-called serious ideals or stimuli of the more earnest storm and stress of life. The stimuli of play are, from the sociological standpoint, comparatively insignificant; the reactions which are building the structure for adult use are vastly the more important part of the function. Play is the propædeutics of the social life and the social is its only justification.

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[1] Fiske, Outlines of Cosmic Philosophy, Vol. 2, pp. 342–369.
Butler, N. M., Anaximander on the Prolongation of Infancy in Man, Classical Studies in Honor of H. Drisler, N. Y., 1894, pp. 8–10.
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Sutherland, Origin and Growth of the Moral Instinct, 2 vols. 1898.
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Vignal, Developpment des éléments du système nerveux cérébro-spinal, Paris, 1889.
Kaes, Archiv fur Psychiatrie, XXV, 1893, pp. 695-758.
Flechsig, Leitungsbahnen im Gehirn und Rueckenmark, Leipzig, 18

Play is instrumental biologically and sociologically in the elimination of instinct and in the increase of social adaptability. Play activities supervene upon and supplant instincts. Social activities in the form of adult ancestral or present day occupations, modified and socialized to fit youthful needs, are grafted on the instinctive impulses. These forms of play produce much greater adjustability to the environment and hence tend to survive in preference to the instincts. The instincts are particular reactions to particular stimuli; play organizes reactions which, in later life, may be attached to numerous objects of ambition. Instincts are, for our practical purposes, invariable. Play may be organized in countless forms to suit the existing social requirements. Instincts are racial and hereditary and reveal the past; play is acquired and educational and is prophetic of the future. Thus the organism, instead of evolving by increase of congenitally inherited adaptations (instincts), evolves much more rapidly through the acquirement of a much greater number of adaptations acquired through play. Civilization is thus made possible through play. Freedom from the incidence of natural selection, as provided for in youth and play, also allows disadvantageous variations arising in the individual's lifetime to drop out and of course permits the increase of the advantageous ones. Thus the increase of parental care during the ages has led to increase of natal weakness, increase of plastic capital, abbreviation of the instincts into impulses, substitution of play reactions for the instincts and thus to the general result of increased adjustability.

The social aim of play is, however, attained largely along the lines determined by the *vis a tergo* of organic and social heredity. Neither phylogeny nor ontogeny set the standards of youthful activities and attainments, yet they are often the guide-posts and furnish the grooves and tracks along which progress takes its way to the ideal. Weak as the human child is through the conditions incident to the law of increase of plastic endowment, he is fashioned in his actions by the instinct-impulses of his organic past and by the stimuli of socially inherited forms of activity. Along these lines he will build according to the architect's plan as laid down by the society in which he lives.

Play is a serious occupation with the child and adolescent. There are moments, it is true, occurring with greater frequency the nearer the adult stage is reached, in which there is a consciousness of the simulation of adult activities; yet, on the whole, play is taken in an objective and business-like way. It is not so much a pretense or a preparation for life to them as it is life itself. The objects of the play-world are as important to them as are our business aims. They also live in a business world. If it were not so there would be no place for them in the so-called serious world of their later adult life. Shielded as they are from the incidence of natural selection, they are nevertheless subject to a natural selection of their own, typical of a struggle yet to come. Mistakes in this preparatory school may be made, and yet a place be still left them for repentance.

Undue influence and interference from the adult-world in this serious play-world may often throw an air of unreality over it, causing the players to live in a world of simulation and engendering confusion and disorder in the growing habits very detrimental in after life. James' statement concerning egoism applies equally to play: "Strong feeling about one's self tends to arrest the free association of one's objective ideas and motor processes."[1] Too much reflection, too much interference, will induce egotistic preoccupation and thus impede the social value of action. The nestling is not the nightingale—why tell him so? "Play," says F. W. Klumpp, "is earnest for the youth. The clever boy conducts his play with a zeal and a devotion of his whole being, such as a man hardly devotes to his most important business."[2]

Too rapid precipitation into adult activities of work and business produces premature mental, moral and physical ossification. The same is of course true in the ordinary school work, owing to the tendency to enormous overpressure due to the accelerated advance of

[1] James' Talks to Teachers on Psychology, p. 219.

[2] In the preface to the 4th ed. of Guts Muths' "Spielen zur Uebung und Erholung des Koerpers und Geistes," quoted by Groos. Groos says: (Spiele der Thiere, p. 69) "That the characteristic difference in the contrast between earnestness and play is the fact that in play, instincts function without earnest stimulus." This, however, seems somewhat defective, inasmuch as one of the terms to be defined enters into the definition just at the point most needful of elucidation. Earnestness or serious occupation and play may after all be characteristic of the same activity. The difference between play and work, for such is the truer contrast, will be alluded to later.

modern civilization and the consequent demands made upon the child to meet that pressure. This work-and-worry attitude may become a habit both in the individual and in the nation, producing weakness, irritability, undue nervous tension, over-conscientiousness and a general lack of finished artistic product. The true artist is the master, and the master is he whose work is play to him, whose work is a matter of delight and joy because of the power and mastery in him over the materials of his occupation. The play-activity must continue through life and become the habit of the nation.

Deprivation of play is one of the chief causes of poverty and degeneracy, inasmuch as the adult who has never played fails to acquire as many of the reactions attainable as he otherwise would, and hence does not possess the *adjustability* requisite for a high position in a society so intricate and complex as it is to-day. A man is, let us say, promoted to be foreman of some works; he lacks habits of careful foresight, prudence, leadership, organization, quick, decisive action in an emergency, etc., which he could and should have acquired in play; hence he fails to meet the needs of his environment and sinks to a lower level. A study of the slum districts of a city substantially confirms these facts and emphatically asserts the need of municipal action in this matter. Play is one of the most important factors of a school curriculum and should be one of the chief cares of any municipality. The recreation-element in play is only a small factor in the enormous problem, although it is generally regarded as the chief, if not only, factor.

Herbert Spencer was, I believe, the first author to attribute the origin of play to instinct.[1] This theory has, however, been carried out much more elaborately by Karl Groos.[2] According to Groos, the play of animals is a manifestation of the instincts peculiar to the species to which the animal in question belongs. It is not only an out-practicing of instincts, but it is also an in-practicing or development of the complete adult instincts. "All plays of youth rest upon instincts. These instincts are not so perfect, not so thoroughly

[1] Spencer, Principles of Psychology, Vol. 2, Chap. on Aesthetic Sentiments.
[2] Karl Groos, The Play of Animals, Appleton, N. Y., 1898. German ed. 1896. Die Spiele der Menschen, Fischer, Jena, 1899.

graven in all details into the brain as they would necessarily have to be in case they were called upon to function in earnest before being trained; they therefore make their appearance already in youth and are thus enabled by means of practice to be developed while there is yet time for preparation." (p. 74.)

"Animals can not be said to play because they are young and frolicsome, but rather they have a period of youth in order to play." "Youth probably exists for the sake of play."[1] The play of young animals, according to Groos, has its origin in the fact that certain very important instincts appear at a time when the animal does not seriously need them. The utility of the exercise of these instincts in the form of play is incalculable, inasmuch as it affords for some of the more important duties of life an invaluable preparation. It paves the way from blind heredity to adaptable and adjustable intelligence. It is especially useful in this respect, since natural selection tends to weaken the force of instinct and aids more and more the development of intelligence as a substitute for it. Thus natural selection favors those animals which play.[2]

The theory of Groos lays too much stress upon instinct. Many play activities are denominated as instinctive by Groos which are really adult ancestral activities handed down by social tradition from generation to generation. Many of them, it is quite true, grow out of or were grafted on to native or congenital impulses; but many again are manifestations of habits acquired *de novo*, the plasticity of

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[1] Groos, The Play of Animals, p. 75.

[2] The plays of animals are classified as follows:

1. Experimentation.
2. Movement plays.
3. Hunting plays:
   - *a.* With real living prey.
   - *b.* With living mock prey.
   - *c.* With lifeless mock prey.
4. Fighting plays:
   - *a.* Teasing.
   - *b.* Tussling among young animals.
   - *c.* Playful fighting among grown animals.
5. Love plays:
   - *a.* Among young animals.
   - *b.* Rhythmical movements.
   - *c.* The display of beautiful and unusual colors and forms.
   - *d.* The production of calls and notes.
   - *e.* The coquetry of the female.
6. Constructive arts.
7. Nursing plays.
8. Imitative plays.
9. Curiosity

the brain allowing the formation of such habits and associations. Many plays are acquired work activities which have gradually become modified into play.[1]

According to the law of increase of plasticity it is manifest that instincts tend to become weakened into impulses, and that as we come up the zoölogical scale instincts tend to become less rigid and organisms to become congenitally weaker and weaker. As the organism becomes weaker it grows in plasticity. As organic heredity weakens, social heredity increases and accumulates. So, in reference to play, it can be easily shown that these adult ancestral occupations and these youthful modifications of them are acquired characteristics, connected though they may be, directly or indirectly, with native impulses.

If play were as Groos defines it, the manifestation (*Ausuebung*) of instinct and the development (*Einuebung*) of the complete instinct, then there would be little if any advance beyond the inheritance provided for us by organic heredity. We should be still limited to the instinctive life of the past. No increase in plasticity or adjustability would result. If, on the other hand, the increase of plastic endowment weakened the instincts to impulses, as it actually does, then *newer and other reactions could be grafted on the inherited impulses.* Here comes in the important rôle of play, *for by means of play social occupational activities are attached to native impulses and take the place of the older instincts.* The young are thus prepared better for life. The rejuvenescence and development of old instincts is thus obviated, at least to some extent. The new reactions would represent greater adjustability and growth in intelligence. In the Groos theory growth in intelligence is claimed, but the *modus operandi* of such growth is not to be found in the development of inherited impulses into their former state as instincts.

Spencer, Stricker, Wundt, Baldwin, Schneider, Groos and others speak of an *instinct* of imitation and an *instinct* of play. To speak of the impulse to imitate and to play is not much improvement on the doctrine. There is no such general instinct or impulse to play.

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[1]Karl Buecher, Arbeit und Rhythmus, Leipzig, 1899, 2te Aufl., *passim.*

Bodily reactions are individual and definite in nature. In each play or imitation certain definite reactions are engaged and others are not. Now, some reactions are acquired and others are inherited. Therefore those cases of play and imitation in which acquired reactions are involved are to that extent at least non-instinctive. Thus there is no general instinct of play or imitation; there are, however, individual or group cases of reaction some of which may be congenital and thus instinctive. There is likewise no general instinct of self-preservation, as usually stated, except in the sense that one name may stand for a number of similar, congenital, concrete reactions.

The error probably arises from the persistent habit of thinking that instinct is a single function of the mind after the manner of the older "faculty" psychology. The error may also arise from the fact that in play there is often a more or less general influx of energy (discharge of surplus-stored energy). This general discharge of surplus energy is, however, a condition of play rather than the play itself, and will be discharged along lines of definite activities, some of which may be instinctive and some acquired.

In a manner similar to the extension of our sense and motor organs by extra-organic means the abbreviated instincts in the shape of impulses are extended by the grafting on of acquired reactions, which have come down to us by means of social heredity. Upon further examination these acquired reactions are found to be profoundly social and occupational in nature. They are usually the survival of the fittest social reactions. Play is thus most intimately connected with social evolution and is social and occupational to the core.[1]

Play activities are either (*1*) *adult, ancestral activities, modified to suit the nature of youth; or,* (*2*) *present-day adult activities, modified to suit the child mind; or,* (*3*) *the usual adult social activities performed with pleasure and the spirit of mastery.*

(I). With progressive civilization ancestral adult experiences tend to be perpetuated and socialized in games and plays. In other

[1] Gambling, waiting for something to turn up, was, in its origin, social and occupational, having been closely connected with divination and religious belief, for which see articles by Cushing.

words, the child tends to recapitulate by social heredity the historical occupations of its remote adult ancestors.

A few examples, though probably well known, may illustrate the wide application and prevalence of this law of social recapitulation:

It seems clear that many of the most popular children's games were originally serious and even solemn ceremonies, which have undergone a gradual process of degradation from their first state, through that of half-joke, half-earnest to their present lowly position. For instance, that well-known terror of the Bank Holiday in England, "Kiss in the Ring," seems to be a relic of the early form of marriage by choice or selection. One of its variants, for there are several ways in which it is played, presents this peculiar feature, that the head of the girl standing in the center of the ring is covered with a shawl, and a portion of the game turns upon her recognition by another player. This indicates, thinks Mrs. Gomme, that "in this game we have preserved one of the ceremonies of a now obsolete marriage-custom—namely, the disguising of the bride and placing her among her bridesmaids and other young girls, all having veils or other coverings alike over their heads and bodies. The bridegroom had to select from among these maidens the girl whom he wished to marry, or whom he had already married, for until this was done he was not allowed to depart with his bride. This custom was continued in sport as one of the ceremonies to be gone through after the marriage was over, long after the custom itself was discontinued. Our bridal veil originates in this custom."[1] A further instance of the complete alteration of character which befalls a custom as it passes through the various stages of its downward evolution, may be studied in the well-known child's song, "Green Gravel," which, little as the children or their mothers suspect it, is, according to the authority just cited, evidently a funeral game. The green gravel and green grass indicate the locality of the scene; "green" as applied to the gravel meaning, probably, freshly disturbed, just as a green grave means a freshly-made grave. The tenant of the newly-made grave is the well-loved lady of a disconsolate lover, and probably the incidents of washing and

[1] Life in Early Britain, Windle, 1897, p. 5.

dressing the corpse, putting an inscription on the place where it is laid, and singing the dirge are indicated in some of the numerous variants of this popular game.[1]

Our marriage ceremony, all-necessary as it is to preserve our social institutions and a high morality, is oftentimes but a merry festival. The "best man" in the village was once needed to effect the capture of a bride, who was one of the most useful and valuable chattels a father possessed. The throwing of rice at a wedding was originally throwing of all sorts of missiles by the enraged relatives of the bride at the successful bridegroom as he carried off his prize.

Grimm informs us that divinities form the core of many of our games and pastimes. The goddess Ostara, or Eástre (Easter), is preserved to us in the name of our Easter festival. Our festivals and games at Easter are probably in their origin rites pertaining to the goddess Ostara. Christianity was wise in grafting on the newer doctrines to the well-established heathen customs and practices. These Easter games came down from dark and remote ages. Easter eggs and the Easter tale, which preachers told from the pulpit for the people's amusement, are old heathen rites.[2] So with the heathen Yule festivities and many other rites.

Fairy tales, with their frequent metamorphoses, may be traced back to god-myths, and even to the earlier stage of the animal-epos. In the myths of primitive America the methods of primitive thought are perhaps better illustrated than in the myths of other countries. Curtin[3] divides the myths of primitive America into two groups, creation myths and action myths. According to the first class there existed at first a world of primitive people or gods, who were different from us entirely. These people were almost innumerable, dwelling together in perfect harmony for an indefinite length of time. The minds of these people, with but a few exceptions, changed, however. Discord broke out, one wanting one thing and another another thing. Conflict set in and the consequence was that in the struggle some were changed into plants, some into animals, others into minerals and

[1] Ibid.
[2] Grimm, *German Mythology*.
[3] Curtin, *Creation Myths of Primitive America*, p. xx.

so on. Thus arose the world as it now is. Creation by metamorphoses seemed a reasonable way to account for the origin of things. After this cycle of myths come the myths which attempt to describe the present world as it now exists with all its happenings. These are the action myths. They describe the conflicts and struggles ensuing upon the creative metamorphoses. Light and darkness, heat and cold, and heroes of all description struggle for the mastery. The value of these myths is that by their means we gain an insight into primitive modes of thought and action. The lives of the first people are described in creation myths and presented as models (early education) upon which faithful Indians fashion their lives at all times and places. The lineal descendants of these and other myths often serve suitably as mental food and appropriate social stimuli for the pupils of to-day.

In the tale of Dornröschen (Thorn-rose or Sleeping Beauty) we have the modern representative of the story of Brünhilde, one of the Walküre, or garden spirits. Just as Dornröschen goes to sleep from the prick of a spindle in the nursery tale, so Brünhilde goes to sleep from the prick of the svefn-thorn or sleep-thorn. Spindles are an essential characteristic of all the wise women of antiquity among Teutons, Celts and Greeks.[1] So in the story of *Jack the Giant-Killer* the hero was originally, I believe, no less than Thor, god of our ancestors.

Local gods also tend to become local saints and sometimes the favorite by-words of merry banqueters and toast-drinkers. They become patron gods and then symbols of the country and its greatness, as, for example, St. George of England. Children still ascribe to their gods exclusive, tribal notions. He is their peculiar god and they are His peculiar chosen people. When Christianity came into contact with the early religion of the Teutons many of the Teuton gods and goddesses were transformed into demons or devils, and their priestesses into witches who possessed great power with these outcast devils.[2] The wise women of the tribes became outcasts living in

[1] Grimm, op. cit., vol. i. p. 419.
[2] Karl Pearson, *The Chances of Death*, "Woman as Witch;" Grimm, *German Mythology*, Vol. iii. p. 986; Otis T. Mason, *Woman's Share in Primitive Culture.*

caves and in the fields. They were surrounded by the domestic animals they had reclaimed from a savage state, such as the goat and the cat, and by the domestic instruments and tools which they had probably invented, such as the broom and the fork.

Another strong instance of this doctrine of social recapitulation is that of the rite of the blood-covenant. It was once undoubtedly a strong social bulwark. Each participant drank of the blood of the other. Incisions were usually made in the arms, the blood was caught, and an eternal compact was completed by each drinking the other's blood. The relationship established was one supposed to be more lasting and closer than that of brotherhood. Gradually, however, wine was substituted for the blood. To-day we drink each other's health as a mere pleasing, after-dinner pastime, forgetful of the life-and-death earnestness of the times in which the custom arose. The ancestral adult experience has become a play or game.

Many occupations of predatory origin and nature give rise to forms of play-activity among the young. Those animals descended from predatory ancestors chase, pursue, bark, bite and fight in a theatrical manner indicative of a much sterner reality in the past. The kitten pursuing a moving string, making it roll and catching it again, crouching as if in ambush, then springing upon its prey, presents to the observer an excellent dramatization of the pursuit and capture of prey. It is an excellent dramatic performance because to the kitten it is quite serious business. In the sport of boys, adult predatory occupations are clearly manifested in their combative games, games of pursuit and capture, struggling, wrestling, teasing, bullying, nagging, etc. Yet these games to-day are socialized exceedingly, inasmuch as they aid materially in the process of social integration. The competitive games of boys, to take one instance out of many, are illustrations of a fundamental law of progress according to which ancestral non-social and possibly anti-social adult activities often tend to survive in the forms of plays as social and integrating forces. The modern football survives because of its encouragement of the social habits of co-operation, inhibition, shrewd-

ness of calculation, celerity of concerted movement, general *corps d'esprit*, etc.

It is, however, hardly necessary to say that all plays are not derived from predatory occupations. There are the plays of courtship, incentive and katabolic, and also occupational plays in which social interaction, group activity, division of labor and rudimentary forms of government are very prominent. One reference alone may be given, taken from the incentive plays of courtship. "In the guayacan, the favorite dance of the Oyampia, men and women form a circle, stamp heavily twice, go forward, let go, embrace each other in couples, and whirl swiftly round in time with the reed-flute. The *bambuko* is nothing but a constant pursuit of the lady; she retires, turns round, at the same time modestly lowering her eyes, lets her arm hang loosely down, hardly raises her foot from the ground, persistently retreats before the charge of her partner, until she at last languidly surrenders and is led away in triumph. If this dance is not an imported fandango, it merely shows what a favorite business all the world over is this story of seeking and winning. Besides this, the Indians of Guiana, who have remained untouched by Spanish and Portuguese influence, like to dance love stories."[1]

Courting itself is a form of combat.[2] The origin of oratory, singing, dancing, combat, rivalry, coquetry, display of form, color and prowess is to be found partly in the courting activities of animals and man. Tattooing, cosmetics and many forms of adornment, including clothing, have a similar origin.[3]

(II). The second form which play tends to assume is that of the present-day adult activities modified to suit the growing plastic mind of youth.

The play of children with dolls revealing, as it does, many characteristic functions of play, may be taken as an example of many

[1] Ratzel, History of Mankind, Vol. II. p. 22.
[2] Colin A. Scott, Sex and Art, Amer. J. of Psych., Vol. 7, No. 2.
[3] Westermarck, History of Human Marriage, London, 1894.
Ratzel, History of Mankind, Vol. I, pp. 195 ff.
Gustav Naumann, Geschlecht und Kunst, Prolegomena zu einer physiologischen Aesthetik, Leipzig, 1899.
Havelock Ellis, Evolution of Modesty, Psych. Rev., March, 1899.

others.[1] Among the Pueblo Indians, the Koreans and Chinese, dolls are exact imitations in miniature of old tribal fetiches or idols no longer worshipped. In the languages of these peoples the word for doll is from the same root as the word for fetich or idol. Doll play as degraded fetich worship is, however, not universal. In the Japanese "Feast of Dolls" the girls play with dolls and toys, mimicking the whole round of Japanese female life, as that of child, maiden, wife, mother and grandmother. Images and effigies are made of the Mikado and his wife, the nobility and the various personages in Japanese history and mythology. To speak of an "instinct of idolatry"[2] is unjustifiable, although it is certainly based on the universal anthropomorphic tendency of mankind.

Doll play is a process of *social dialectic*, possessing certain advantages over the usual intercourse of the child with living playmates. A rudimentary sociology, ethics and science is evidenced which is not always attained by playing with their mates. Children with French dolls practice their little French upon them, read stories to them, tell them their private griefs and ailments, are good to set them a good example, live in a social world of rights and duties with them, etc. The doll world is a miniature adult world. It aids the children in motor expression and thus forms a useful antidote to over-didacticism and excessive cephalization so prevalent in the grades.

From the returns of Ellis and Hall many opinions on the influence of dolls on children may be cited: they are manifestations of nascent parenthood, they cultivate the imagination, train children for domestic life, develop the moral qualities, cultivate taste in dress, teach to sew, teach tidiness, thoroughness, imitation is stimulated, they develop more thoughtfulness for and sympathy with others, keep out of mischief, keep them busy, "keep children from growing old;" "best of all," according to one, "is the reflex influence on the child of trying to teach her doll and of trying to set a good example."

Many other instances of imitations of adult occupations could be cited, such as nursing, playing horse, building a house, etc., but

[1] Ellis and Hall, A Study of Dolls, Pedagogical Seminary, Oct. 1896, from which some of these data are taken.
[2] Ib. p. 174

they are too familiar to need mention. It has also been pointed out that the games of the Roman people adumbrated the principles of their civilization. This, too, was the case among the Greeks.[1]

(III). The third form which play tends to assume is that of the usual social adult occupations performed with pleasure and in the spirit of mastery.

All traditional forms of play are not play to those acting in them. Under certain circumstances they may be matters of hard, joyless work. Moreover, some forms of what is ordinarily called work are often play. In short, play is characterized not by the original form of activity, but by the attitude of the mind. Earnestness and business-like attention to the matter in hand are characteristics of play as well as of what is ordinarily called work. The spirit of joyful mastery is the preeminent spirit of play. Thus regarded, one's occupation or profession may be play. Joyless toil is the true antithesis of play. The physicist, for example, who has his whole heart and soul in his occupation, who delights in his work, who spends his nights and days willingly in the solution of perplexing problems, regards his work as play. He follows his imperious interests, is a slave to his ruling inclinations and feels himself a joyful master in a joyful subject. A truly religious man is he who cannot help doing good for the very love of it. He has the Aristotelian ἕξις rather than the Kantian conscience. The inexorable categorical imperative has become an overmastering inclination. The Mosaic Ten Commandments are more than fulfilled by that greatest of all inclinations—love. The opposite of play is toil—the *via dolorosa* with nothing but a cross in view.

Viewed in this way we see that play and art are synonymous; that is, when art is regarded from the subjective standpoint, the standpoint of the doer and maker. Regarded from the subjective standpoint art is doing the best one can under the highest motives of which the doer is capable and with a spirit of joyful mastery. Art is the feeling of best-doing in every way. The best may be a daub, a blotch, a shapeless mass of clay, a discordant cry, but it is

[1] W. T. Harris, Psychologic Foundations of Education, pp. 283-286.

art if it is the best. "Art," says William Morris, "is the expression of man's joy in his work; the curse of the world is joyless labor." Take for instance the drawings of childhood. They are bold creations of a dauntless spirit, *insouciant*, evidences of the simple faith in one's self, of a delightful disregard of law and order and of a mind deeply intent, overwhelmingly intent, on expressing facts. To discourage the work of these young artists is to nip art in the bud.

Much confusion in the psychology of play and art has resulted from the confusion of two standpoints. Art when looked at from the subjective standpoint may be quite a different thing from art looked at from the objective standpoint. A recluse of an inventor may produce a machine marvelous for its intricacy and complexity, and from the subjective standpoint he may be an artist in his work; as an expression of thought and feeling, as an expression of a social judgment, as a public utility, the product of his activity may not be regarded as a work of art. A painting may appeal to an anachronistic instinct and the painter may rejoice in his work and in the success of his appeal, but from the objective standpoint the painting may not be regarded as artistic. In the school room prescribed drawing, painting or music may be an abomination of desolation to a pupil from the subjective standpoint. Looked at from the subjective standpoint we study the activity of production; looked at from the objective, social standpoint we study the product.

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APPLICATIONS OF ELLIPTIC FUNCTIONS TO PROBLEMS OF CLOSURE

By Arnold Emch

INTRODUCTION

Elliptic functions as a special branch of the theory of functions occupy an extremely important place in modern mathematics. Similar interest is attached to them with reference to their beautiful applications in geometry and mechanics. As a result of the investigations along these lines we have the valuable treatises by Greenhill, Halphen, Appell and Lacour, and a great number of monographs distributed in various mathematical journals. Other treatises on elliptic functions, with a few exceptions, invariably contain a chapter on the applications.

In this article I shall make a limited collection of problems concerning variable figures with the property of closing in every particular position.

Naturally many of the problems to be treated are well known in one or another form, while on the other hand, in the last two chapters, I shall add some of my own investigations on this subject, partly new, partly already published.

The first chapter treats of Abel's theorem and its application to plane curves of the third order. Here the principal object is to bring out Steiner's celebrated problems of closure (Schliessungsprobleme) by the method first established by Clebsch.

The second chapter contains a generalization of Abel's theorem and its special application to twisted curves of the fourth order of the first kind and the pencil of quadrics passing through it. A sketch of the constructive treatment of the same problems concludes the chapter.

In the third chapter I shall show how elliptic functions make their appearance in the theory of certain linkages. Poncelet's polygons and Steiner's circular series result from this theory as special cases.

Loxodromics on the torus, and in general on Dupin's cyclides form the subject of the fourth and last chapter.

In the first two chapters I started from the most general propositions and passed through the intermediate steps which are necessary to show in a connected manner how the Abelian theorem enters into the theory of elliptic functions and their geometrical applications.

I. ABEL'S THEOREM. APPLICATION OF ELLIPTIC FUNCTIONS TO PLANE CUBICS. PROBLEMS OF CLOSURE

§1. Abel's Theorem[1].

1. Let

$$f(x, y)=0 \tag{1}$$

be the equation of an algebraic curve Cm of the mth order and

$$v(x, y)=\int_{(x_0, y_0)}^{(x, y)} \mathrm{R}(x, y)\, dx \tag{2}$$

any Abelian integral attached to this curve. Consider a system of algebraic curves

$$\phi(x, y, a_1, a_2, \ldots, a_r)=0 \tag{3}$$

of the nth order depending upon r arbitrary parameters $a_1, a_2, \ldots a_r$ which are supposed to be contained rationally in (3). The curve (1) and each curve of (3) have $\mu=mn$ points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_\mu, y_\mu)$$

(1) For a complete treatment of this theorem see Picard: Traité d'Analyse. Vol. II, pp. 393-396. Appell et Coursat: Théorie des Fonctions Algébriques, pp. 401-403, which I have here followed more or less.

in common which vary with the arbitrary parameters. The sum of the values of the Abelian integral $v\ (x, y)$, in which each point of intersection is sucessively taken as the upper limit.

$$S = v(x_1, y_1) + \ldots + v(x_\mu, y_\mu) = \sum_{i=1}^{\mu} \int_{(x_0, y_0)}^{(x_i\ y_i)} R\ (x, y)\ dx \qquad (4)$$

is a function of the parameters a. It is proposed to find the analytic form of this function.

The quantities $x_1, x_2, \ldots, x_\mu$ are roots of a certain equation.

$$\lambda\ (x, a_1, a_2, \ldots, a_r) = 0 \qquad (5)$$

of degree μ whose coefficients are rational with respect to the parameters a. If the curves (1) and (3) have no particular position with respect to the axes it is always admissible to assume the corresponding value of y in the form.

$$y = \psi\ (x, a_1, a_2, \ldots, a_r). \qquad (6)$$

Designate now by δ V the total differential of a function V with respect to the parameters $a_1, a_2, \ldots, a_r$, then

$$\delta S = R\ (x_1, y_1)\ \delta x_1 + \ldots + R\ (x_\mu, y_\mu)\ \delta x_n. \qquad (7)$$

Differentiating relation (6), one can calculate in succession δx_1, δx_2, $\ldots$, $\delta x\mu$ and substitute in the expression (7). Replacing the y's by their values ψ, one has for the coefficient of δa_1 a rational function of $x_1, x_2, \ldots, x_\mu$ and of the a's. It is moreover symmetrical with respect to the x's and consequently the coefficient of δa_1 is a rational function of the a's. Similar relations hold for the other coefficients. Hence

$$\delta S = P_1\ (a_1, \ldots, a_r)\ \delta a_1 + P_2\ (a_1, \ldots, a_r)\ \delta a_2 + \ldots + P_r\ (a_1, \ldots, a_r)\ \delta a_r,$$

where the P's are rational functions of $a_1, a_2, \ldots, a_r$, and

$$S = \int P_1\ \delta a_1 + P_2\ \delta a_2 + \ldots + P_r\ \delta a_r. \qquad (8)$$

The integration cannot introduce any other transcendental functions

than logarithms; hence *the sum* S *is a rational function of the coefficients* $a_1, a_2, \ldots a_r$, *increased by a sum of logarithms of rational functions of the same coefficients:*

$$S = \rho + \Sigma A \log \sigma, \tag{9}$$

the A*'s being constants.*

2. We shall specialise this theorem for the case of Abelian integrals of the first kind. Such an integral

$$\int_{(x_0, y_0)}^{(x, y)} R(x, y)\, dx$$

remains finite for every point (x, y) of the corresponding Riemann surface. The function (9), which represents S, must remain finite for all finite or infinite values of the parameters a. But a function

$$\rho(a) + \Sigma A \log \sigma(a),$$

where $\rho(a)$ and $\sigma(a)$ are rational functions of a, which remains finite for all values of a, reduces necessarily to a constant.

Hence, *in case of an Abelian integral of the first kind, the sum*

$$\sum_{n=1}^{n=\mu} \int_{(x_0, y_0)}^{(x_n, y_n)} R(x, y)\, dx, \tag{10}$$

where (x_n, y_n) *designates the points of intersection of the curves* (1) *and* (3), *which vary with the a's, is independent of these parameters and* (*excepting sums of multiples of certain fixed periods which may always be introduced*) *has a constant value.*

3. The Abelian integrals of the first kind with respect to a curve

$$f(x, y) = 0$$

of the mth order are, as is well known, of the form

$$\int \frac{Q(x, y)\, dx}{\frac{\delta f^1}{\delta y}}, \tag{11}$$

δ is here and subsequently used as a partial differential sign.

with the necessary and sufficient condition that $Q(x, y)$ be a polynomial of degree $m-3$ in x and y, and that the curve

$$Q(x, y)=0$$

shall have as multiple points of order $i-1$ the multiple points of order i of the curve $f(x, y)=0$[1]. Replacing the integral in (10) by (11) and differentiating, Abel's specialised theorem may also be written in the form

$$\sum_{n=1}^{n=\mu} \frac{Q(x_n, y_n)\, dx_n}{f'_y(x_n, y_n)}=0. \tag{12}$$

§2. Intersection of a Cubic by a Straight Line, and Generally by a Curve of the nth Order.

1. If in formula (12) $f(x, y)=0$ represents a plane cubic, then $Q(x, y)$ is a constant and Abel's theorem becomes

$$\sum_{n=1}^{n=\mu} \frac{dx_n}{f'_y(x_n, y_n)}=0. \tag{13}$$

Let the variable curve (3) be a straight line, then

$$\frac{dx_1}{f'_y(x_1, y_1)}+\frac{dx_2}{f'_y(x_2, y_2)}+\frac{dx_3}{f'_y(x_3, y_3)}=0, \tag{14}$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3) represent the points of intersection of the straight line with the cubic, and assume the equation of the cubic in the form

$$y^2=4x^3-g_2x-g_3. \tag{15}$$

From analytic geometry it is known that the equation of every cubic may be reduced to this form by collineation, so that by assuming (15) nothing is lost in generality.

Establish now the elliptic function $p^2(u, g_2, g_3)$, then from the well known differential equation

---

(1) Picard, loc. cit., p. 401.

(2) Here and in the following sections where the Weierstrassian sign for elliptic functions is needed, the character $\mathcal{P}$ shall be used in its place.

$$p'^2 u = 4p^3 u - g_2 p u - g_3$$

it is immediately seen that

$$x = pu,\ y = p'u \tag{16}$$

satisfy (15). To every value of u corresponds a point of the curve, since p and p' are uniform and the point (x, y) remains the same if the argument u is increased by multiples of the periods. Conversely, to a point of the curve corresponds one and only one value of u. The application of (16) to (14) gives, since

$$dx = p'(u)\,du,\ or\ \frac{dx}{dy} = du,$$

$$\text{and from (15)}\ \frac{\delta f}{\delta y} = 2y, \frac{dx}{2y} = \frac{dx}{f'_y(x, y)} = \frac{du}{2},$$

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} = 0,$$

or designating the values of u corresponding to the three points of intersection by u_1, u_2, u_3,

$$du_1 + du_2 + du_3 = 0,$$

or

$$u_1 + u_2 + u_3 = const. \tag{17}$$

Hence:

The sum of the arguments corresponding to the three points of intersection of a straight line with a cubic is constant.

To determine the value of this constant assume the x-axis as a particular position of the variable straight line. Then $p'u = 0$. The values of u for which this is the case, are the half-periods w, w_1, and $w + w_1$, consequently also the values $w + 2kw + 2k_1w_1$, $w_1 + 2kw + 2k_1w_1$, $w + w_1 + 2kw + 2k_1w_1$.

Hence, designating by l and l_1 new integers,

$$u_1 + u_2 + u_3 = 2w + 2w_1 + 6kw + 6k_1w_1 = 2lw + 2l_1w_1,$$

or

$$u_1 + u_2 + u_3 \equiv 0 \text{ (mod. per.)}^1. \tag{18}$$

(1) Following this $\equiv o$ shall always signify $\equiv o$ (modulus periodicity).

2. To make this result familiar we shall apply it to the determination of the points of inflexion of a cubic and show its importance for the solution of problems of closure afterwards.

The tangent at a point of inflexion intersects the curve in three consecutive points. Hence, if u is the argument of such a point,

$$u_1=u_2=u_3=u,$$

from which

$$u=\frac{2lw+2l_1w_1}{3}. \tag{19}$$

In this expression l and l_1 may be given all integral values; but two values of u which differ only by multiples of $2w$ and $2w_1$ do not give different points of inflexion. It is therefore sufficient to give l and l_1 the values 0, 1, 2, associated in all possible ways. The cubic has therefore 9 points of inflexion, and their arguments are, indicating the values associated to l and l_1 in every case:

$$\begin{gathered}
u_{0\,0}=0\ u_{0\,1}=\frac{2w_1}{3},\ u_{0\,2}=\frac{4w_1}{3},\\
u_{1\,0}=\frac{2w}{3},\ u_{1\,1}=\frac{2w+4w_1}{3},\ u_{1\,2}=\frac{2w+2w_1}{3},\\
u_{2\,0}=\frac{4w}{3},\ u_{2\,1}=\frac{4w+2w_1}{3},\ u_{2\,2}=\frac{4w+4w_1}{3}.
\end{gathered} \tag{20}$$

These points are three by three in straight lines; the straight line joining two always passes through a third. For example

$$u_{0\,0}+u_{1\,1}+u_{2\,2}=2w+2w_1$$

shows, according to (18), that the corresponding points of inflexion are collinear. As $p'(o)=\infty$, it follows that the first point of inflexion $u_{0\,0}$ is infinitely distant in the direction of the y-axis.

3. If from a point u_1 a tangent is drawn to the cubic, then we have, since $u_2=u_3=x$ (to be found),

$$2x+u_1=2mw+2m_1w_1,$$

and

$$x = -\frac{u_1}{2} + \frac{2mw + 2m_1w_1}{2},$$

where m and m_1 can assume the values 0 and 1. This gives for the arguments of the points of tangency of the four tangents from a point u of the cubic

$$-\frac{u}{2}, -\frac{u+2w}{2}, -\frac{u+2w_1}{2}, -\frac{u+2w+2w_1}{2}. \tag{21}$$

If we take for u a point of inflexion $\frac{2mw+2m_1w_1}{3}$, the arguments of the points of tangency become

$$-\frac{2mw+2m_1w_1}{6}, -\frac{(m+3)2w+2m_1w_1}{6}, -\frac{2mw+(m_1+3)2w_1}{6},$$

$$-\frac{(m+3)2w+(m_1+3)2w_1}{6}. \tag{22}$$

4. The theorem contained in formula (18) may be generalized.

A curve $C_n=0$ of the nth order cuts the cubic in $3n$ points. $3n-1$ points may be chosen arbitrarily, and *these determine the remaining point.* To prove this it is to be remembered that the equation $C_n=0$ contains $\frac{(n+1)(n+2)}{2}$ arbitrary coefficients in a linear and homogeneous combination. The $3n$ points of intersection are not changed if the curve $C_n=0$ is replaced by a curve with the equation.

$$C'_n \equiv C_n - f(x, y)\, C_{n-3} = 0,$$

where C_{n-3} is a polynomial of degree $n-3$. As this polynomial contains $\frac{(n-2)(n-1)}{2}$ arbitrary coefficients it is possible to dispose of these in such a manner that $\frac{(n-2)(n-1)}{2}$ terms in C'_n disappear, so that only

$$\frac{(n+1)(n+2)}{2} - \frac{(n-2)(n-1)}{2} = 3n$$

arbitrary coefficients remain in C'_n in a linear and homogeneous form. These coefficients may be chosen in such a manner that the curve $C'_n=0$ passes through $3n-1$ points on the cubic $f(x, y)=0$, so that the remaining point of intersection is entirely determined. Hence one and only one relation can exist between the values $u_1, u_2, \ldots, u_{3n}$ of the parameter u corresponding to the $2u$ points of intersection. Hence, by Abels theorem

$$u_1+u_2+\ldots+u_{3n}\equiv \text{const. (mod. per.)}$$

5. In case of a conic

$$u_1+u_2+u_3+u_4+u_5+u_6\equiv 0.$$

The constant may be neglected without loss of generality.

The conic has a contact of the 6th order with the cubic if $u_1=u_2=\ldots=u_6$, i. e., if

$$u=\frac{mw+m_1w_1}{3}.$$

Here m and m_1 can take all values from 0 to 5, which gives 36 points. Among these the 9 points of inflexion are included, so that in reality only $6^2-3^2=27$ points of contact of hyperosculating conics exist on a cubic. These points are six by six on conics, for example

$$u_{0\,0}=0$$
$$u_{1\,1}=\frac{w+w_1}{3}$$
$$u_{2\,2}=\frac{2w+2w_1}{3}$$
$$u_{3\,3}=\frac{3w+3w_1}{3}$$
$$u_{4\,1}=\frac{4w+w_1}{3}$$
$$u_{2\,5}=\frac{2w+5w_1}{3},$$

the sum of whose arguments is $4w+4w_1$.

§3. Problems of Closure on the Plane Cubic.

1. In 1845 Steiner published an important memoir, "Geometrische Lehrsätze", in Crelle's Journal, Vol. 32, pp. 371-373, containing the now famous problem on closing polygons on a cubic. Since that time a great number of geometers have studied these problems[1]. Of all these Clebsch[2] has probably made the most important contribution to the subject. From a purely geometrical stand-point the work of Desteli, "Die Steinerschen Schliessungsprobleme nach darstellend geometrischer Methode"[3] ranks undoubtedly among the best contributions to the subject.

2. We shall first establish Steiner's theorem in its generalized form.

Assume n distinct arbitrary points on a plane cubic, with the arguments $v_1, v_2, \ldots, v_n$. For the sake of simplicity designate the points themselves by the same symbols. Through v_1 pass any straight line cutting the cubic in two other points with the arguments u_1 and u_2 (designate these points by their arguments as before and hereafter).

Through u_2 and v_2 pass another straight line, cutting the cubic at u_3; through u_3 and v_3 pass a line, cutting the cubic at u_4; and so forth; finally connect u_n with v_n, cutting the cubic at u_{n+1}. *The problem is to find the condition for which u_{n+1} is identical with u_1, no matter how u_1 may have been chosen.* We evidently have:—

$$\begin{aligned}
&1. \quad u_1+v_1+u_2\equiv 0,\\
&2. \quad u_2+v_2+u_3\equiv 0,\\
&3. \quad u_3+v_3+u_4\equiv 0,\\
&\qquad\vdots\\
&n-1. \quad u_{n-1}+v_{n-1}+u_n\equiv 0,\\
&n. \quad u_n+v_n+u_{n+1}\equiv 0.
\end{aligned} \tag{23}$$

[1] The reader is referred to the interesting and valuable pamphlet of Prof. Gino Loria: "I Poligoni di Poncelet, discorso pronunziato nell' Universitia di Genova. Turin, 1889, and to the list of references at the end."

[2] (*a*) Über einen Satz von Steiner und einige Punkte der Theorie der Curven dritter Ordnung. Crelle's Journal, Vol. 63, 1864, pp. 94-121.
(*b*) Vorlesnugen über Geometrie, Vol. I, pp. 615-627.

[3] Teubner, Leipzig, 1888.

The two cases must be distinguished where n is odd or even. If n is odd and adding in (23) the congruences with odd indices, then subtracting from this sum the sum of the congruences with even indices we obtain:—

$$u_1+u_{n+1}\equiv\sum_{1}^{\frac{n-1}{2}}v_{2k}-\sum_{0}^{\frac{n-1}{2}}v_{2k+1}.$$

In this congruence it is evidently impossible to make $u_{n+1}=u_1$, for any value of u_1. In other words the problem cannot be solved for an odd n.

In a similar manner we obtain for our even n:

$$u-u_{n+1}\equiv\sum_{1}^{\frac{u}{2}}v_{2k}-\sum_{0}^{\frac{n-2}{2}}v_{2k+1},$$

which for $u_{n+1}=u_1$ becomes:—

$$\sum_{1}^{\frac{n}{2}}v_{2k}-\sum_{0}^{\frac{n-2}{2}}v_{2k+1}\equiv 0, \qquad (24)$$

a congruence which can always be satisfied by choosing the v's properly. If $n-1$ of the v's are arbitrarily chosen, the nth v is by (24) uniquely determined. Hence the theorem:—

The vertices of a polygon of 2n sides, $A_1\ A_2\ \ldots\ A_{2n}$, *may be moved on a plane cubic in such a manner that all its sides* A_1A_2, A_2A_3, $\ldots$, $A_{2n}A_1$ *turn about fixed points of this curve. One of these fixed points is uniquely determined by all the others.*

From (23) it is easily found that:—

$$u_1+u_4+(v_1+v_3-v_2)\equiv 0,$$
$$u_2+u_5+(v_2+v_4-v_3)\equiv 0,$$
$$\text{etc.}$$

Hence the corollary:

Also the sides A_1A_4, A_1A_6, $\ldots$, A_1A_{2n}; A_2A_5, A_2A_7, $\ldots$; $A_{2n-3}A_{2n}$ *turn about fixed points of the curve.*[1]

(1) See KÖTTER: Die Entwickelung der synthetischen Geometrie, Vol. I, p. 153.

3. *Steiner's Theorem.* On a cubic C_3 take two points v_1 and v_2. Through v_1 draw a straight line cutting C_3 in u_1 and u_2. Through u_2 and v_2 draw a line cutting C_3 in u_3; again through u_3 and v_1 draw a line cutting C_3 in u_4; and so forth. The problem is to locate v_2 in such a manner that the $2n$th side passes through u_1. The conditions evidently are:

$$\begin{aligned} v_1+u_1+u_2 &\equiv 0, & v_2+u_2+u_3 &\equiv 0,\\ v_1+u_3+u_4 &\equiv 0, & v_2+u_4+u_5 &\equiv 0,\\ v_1+u_5+u_6 &\equiv 0, & v_2+u_6+u_7 &\equiv 0,\\ &\cdots & &\cdots\\ v_1+u_{2n-1}+u_{2n} &\equiv 0, & v_2+u_{2n}+u_1 &\equiv 0. \end{aligned}$$

Adding

$$nv_1+\Sigma u \equiv 0, \qquad nv_2+\Sigma u \equiv 0,$$

or

$$v_1=v_2+\frac{1}{n}(2mw+2m_1w_1). \tag{25}$$

Hence, if v_1 is given, v_2 is determined many-valued. If one of these values of v_2 is known from (25), all vertices of the polygon may be determined as soon as one, say u_1, has been arbitrarily assumed. This result may be stated in the theorem, due to Steiner:

The vertices of a polygon of 2n sides may be moved on a cubic in such a manner that its odd sides pass through one fixed point v_1, while all the even sides pass through a certain fixed point v_2.

4. *A Problem of Closure admitting of a limited number of solutions.*

From any point u_1 of a cubic draw a tangent to it and let u_2 be the point of tangency; from u_2 draw a new tangent with the point of tangency u_3, and so forth. In this manner a series of points

$$u_1, u_2, u_3, \ldots, u_{n+1}$$

is obtained. The problem is to find the condition that u_{n+1} coincides

with u_1. Designating multiples of periods by $A_1, A_2, A_3, \ldots, A_n$ we have:—

$$\begin{aligned} u_1+2u_2&=A_1,\\ u_2+2u_3&=A_2,\\ u_3+2u_4&=A_3,\\ &\cdot\\ &\cdot\\ &\cdot\\ u_{n-1}+2u_n&=A_{n-1},\\ u_n+2u_{n+1}&=A_n, \end{aligned}$$

from which, if $u_{n+1}=u_1$

$$\begin{aligned} u_n&=A_n-2u_1,\\ u_{n-1}&=A_{n-1}-2u_n=A_{n-1}-2A_n+2^2u_1,\\ u_{n-2}&=A_{n-2}-2u_{n-1}=A_{n-2}-2A_{n-1}+4A_n-2^3u_1,\\ &\cdot\\ &\cdot\\ &\cdot\\ u_1&=2mw+2m_1w_1+(-1)^n2^nu_1, \end{aligned}$$

and

$$u_1=\frac{2mw+2m_1w_1}{(-1)^{n-1}2^n+1}. \tag{26}$$

From this it is seen that in every case there are a limited number of solutions which depend upon the division-problem of elliptic functions. For $n=1$ the points of inflection are obtained. For $n=3$

$$u_1=\frac{mw2+2m_1w_1}{9},$$

in which m and m_1 may assume all possible values from 0 to 8. Among the 81 solutions are contained the 9 points of inflexion, so that only 72 ordinary closed triangles of the prescribed kind exist on the cubic. We have here an essentially different problem from those with an infinite number of solutions.

§4. Steiner's Couples and their Properties.

1. In §3 the equation

$$v_1 = v_2 + \frac{1}{n}(2mw + 2m_1w_1) \qquad (27)$$

has been established, by which two points v_1 and v_2 on a cubic must be related in order to serve as fixed points of an infinite number of closed polygons. Any two points of this kind form a *Steinerian couple.*

To find the coördinates of v_1 when v_2 is given, say $v_2 = \frac{a}{n}$, by (16)

$$x = p\left(\frac{a + 2mw + 2m_1w_1}{n}\right). \qquad (28)$$

The value of $p\left(\frac{a}{n}\right)$ being given, x may be expressed in terms of this in n^2 different ways, as is well known from the division-problem of elliptic functions. Of these we have to exclude the case where $v_1 = v_2$, so that in reality only $n^2 - 1$ pairs are left. It is clear that this number is reduced when n is not an odd prime. The result may be stated in the theorem:—

To every point v_2 of the curve are associated as many points v_1, forming in each case a couple, as there are pairs of numbers m, m_1, $(<n)$ which have no common factor with n.

From formulas (20) and (27) the special theorem is evident:—

Any two points of inflexion of a cubic may be the fixed points of a Steinerian polygon of six sides.

Taking the points of tangency of the tangents from any two points of inflexion and applying formulas (22) it is found that *these points of tangency are the fixed points of a Steinerian polygon of twelve sides.*

2. All couples for which m and m_1 have constant values are said to belong to the same class, which is characterized by the combination m, m_1. To obtain all couples of the same class, when one (v_1, v_2) is known, connect any point v of the curve with v_1 and v_2 by straight lines which cut the curve in two other points v_1' and v_2', respectively. Then

$$v_1+v+v_1' \equiv 0,$$
$$v_2+v+v_2' \equiv 0,$$

and

$$v_1-v_2 \equiv v_1'-v_2'. \tag{29}$$

Hence v_1' and v_2' form a couple of the same class, and we have the theorem:—

The straight lines joining the points of a Steinerian couple with any point of the curve cut the curve in a new couple of the same class; all couples of the same class are obtained by letting v describe the whole curve.

The case where $n=2$, i. e., of quadrilaterals, is of particular interest. Formula (27) now becomes

$$v_2=v_1-\tfrac{1}{2}\,(2mw+2m_1w_1),$$

in which m and m_1 may assume the values 0 and 1. The original point together with the possibilities $(0,1)$; $(1,0)$; $(1,1)$ gives the four points

$$v_1,\ v_1-w_1,\ v_1-w,\ v_1-(w+w_1). \tag{30}$$

Substituting any of these arguments in (27) it is seen that *any two of these points form a Steinerian couple.*

Four points with this property may be called a *Steinerian quadruple.*

If at a point u of a cubic a tangent is drawn, $2u+x \equiv 0$, hence

$$x \equiv -2u. \tag{31}$$

Substituting the values of (30) in (31), it is found that *the tangents at the points of a quadruple are concurrent.*

The rays joining a point v' of the curve with the quadruple (30), or any other quadruple as it may easily be verified, intersect the curve in another quadruple whose arguments in this particular case are

$$\begin{aligned} v_1' &\equiv -v_1-v', \\ v_2' &\equiv -v_1-v'+w, \\ v_3' &\equiv -v_1-v'+w_1, \\ v_4' &\equiv -v_1-v'+w+w_1. \end{aligned} \tag{32}$$

Taking a second quadruple

$$\begin{aligned} v_1'' &\equiv -v_1 - v'', \\ v_2'' &\equiv -v_1 - v'' + w, \\ v_3'' &\equiv -v_1 - v'' + w_1, \\ v_4'' &\equiv -v_1 - v'' + w + w_1, \end{aligned} \tag{33}$$

then the lines joining v_1' with v_1'', v_2'', v_3'', v_4'' cut the curve in four points

$$\begin{aligned} x_1 &\equiv 2v_1 + v' + v'', \\ x_2 &\equiv 2v_1 + v' + v'' - w, \\ x_3 &\equiv 2v_1 + v' + v'' - w_1, \\ x_4 &\equiv 2v_1 + v' + v'' - w - w_1. \end{aligned} \tag{34}$$

Making the same construction for each of the points v_2', v_3', v_4', and the four points of (33), the same points of intersection x_1, x_2, x_3, x_4 are obtained in every case. Thus the theorems of Clebsch, loc. cit. (a), follow immediately: —

The straight lines joining any point of a cubic with a Steinerian quadruple meet the curve in a second quadruple. From a single quadruple all others are obtained by letting the variable point describe the whole curve.

The sixteen lines joining the points of any two quadruples intersect each other four by four in four points of a third quadruple.

In §§8, 9 it will be seen what connection these theorems have with the twisted curve of the fourth order and of the first species.

§5. Case of Degenerating Cubic.

Suppose the cubic degenerates into a straight line and a conic (in Figs. 1, 2, 3, a circle). Steiner's generalized and special theorems still hold. Thus, a great number of theorems may be derived of which only three shall be mentioned. From the previous considerations the proofs for these are evident. On the other hand they may be easily derived from well-known projective properties.

1. *If a quadrilateral $u_1u_2u_3u_4$ is inscribed in a circle, Fig. 1, and its sides by any straight line l are cut in the four points*

v_1, v_2, v_3, v_4, then there are an infinite number of quadrilaterals, $u_1'u_2'u_3'u_4'$, $u_1''u_2''u_3''u_4''$, . . . , inscribed into the same circle with their sides similarly passing through v_1, v_2, v_3, v_4. Corresponding diagonals pass through two fixed points of l.

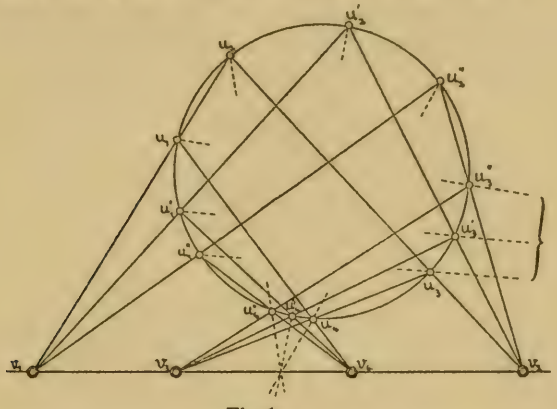


Fig. 1.

2. *If a quadrilateral has two of its vertices, u_1, u_3, on a circle and the two others, u_2, u_4, on a straight line and if its sides, u_1u_2, u_2u_3, u_3u_4, u_4u_1, pass through the points, v_1, v_2, v_3, v_4 of the circle, Fig. 2, then there are an infinite number of quadrilaterals, $u_1'u_2'u_3'u_4'$, $u_1''u_2''u_3''u_4''$, . . . having in a similar manner two vertices on the given circle and two on the given line, and their sides passing through the fixed points, v_1, v_2, v_3, v_4.*

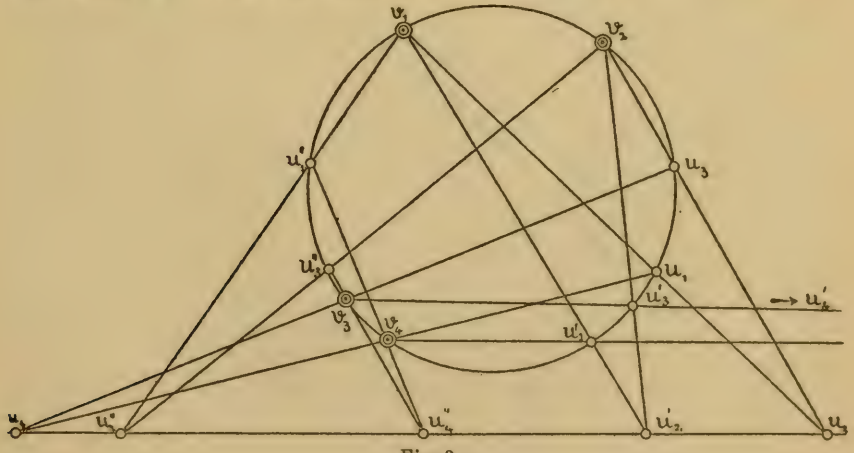


Fig. 2.

3. If the sides of a quadrilateral $u_1u_2u_3u_4$, inscribed to a circle, alternately pass through two fixed points v_1, v_2 of a straight line l, Fig. 3, then there are an infinite number of inscribed quadrilaterals $u_1'u_2'u_3'u_4'$, $u_1''u_2''u_3''u_4''$, . . . with the same property. The corresponding diagonals of all these quadrilaterals pass through a fixed point, the pole of v_1v_2.

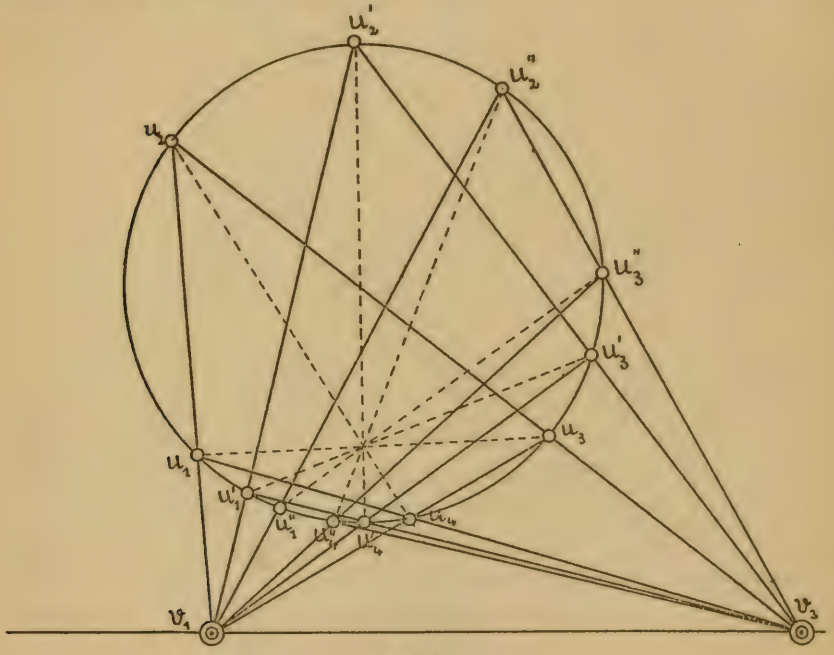


Fig. 3.

These theorems, to which the reader may add a great number of others, may, of course, immediately be duplicated by the principle of duality. It is well to remark that the first and third of these theorems are also special cases of the problem, *to inscribe a polygon to a conic whose sides, in a certain order, pass through fixed points of the plane*, which in case of a triangle and circle was already solved by Castillion and Lagrange in 1776. The extension of the problem to polygons was given by Oltajano and Malfatti. For the triangle the extension to conics was given by Brianchon and

GERGONNE. The solution of the general problem was given by PONCELET in his "Traité des propriétés projectives," p. 352, 2d ed., p. 340. For these references see FIEDLER-SALMON's Analyt. Geometrie der Kegelschnitte, ii, p. xii, (88).

II. PROBLEMS OF CLOSURE ON THE TWISTED QUARTIC OF THE FIRST SPECIES. PROJECTIVE THEORY.

§6. GENERALIZATION OF ABEL'S THEOREM.[1]

1. In order to extend Abel's theorem to twisted curves, assume a plane curve C of deficiency p with the equation

$$f(x, y)=0, \tag{1}$$

and a point with the coördinates

$$X=\phi(x, y),\ Y=\psi(x, y),\ Z=\chi(x, y), \tag{2}$$

where ϕ, ψ, χ are rational functions of x and y. If x and y vary according to (1), the point (2) describes a twisted curve Γ which corresponds point for point to the curve C, provided ϕ, ψ, χ have not been chosen in some particular manner, which is supposed to be excluded. Hence, x and y are, inversely, rational functions of X, Y, Z. Every integral of the form

$$\int \alpha dX+\beta dY+\gamma dZ \tag{3}$$

taken along Γ, where α, β, γ are rational functions of X, Y, Z, by substitution (2), is transformed into an Abelian integral

$$\int R(x, y)dx$$

with respect to the curve C. On the other hand let

$$\Pi(X, Y, Z)=\frac{P(X, Y, Z)}{Q(X, Y, Z)} \tag{4}$$

be a rational function, P and Q being of the same degree. Replacing X, Y, Z by ϕ, ψ, χ, we have

(1) See APPELL et COURSAT, loc. cit., p. 432-434.

$$\Pi(X, Y, Z)=\Pi_1(x, y), \tag{5}$$

Π_1 being a rational function of the point (x, y) whose zeros correspond to the points of intersection of the curve Γ with the surface S having the equation $P(X, Y, Z)=0$, and the poles to the points of intersection with the surface S′ represented by $Q(X, Y, Z)=0$. Hence, by Abel's theorem, it is possible to express by algebraic and logarithmic quantities the difference between the sum of the values of integral (3) at the points of intersection of the curve Γ and the surface S, and the sum at the points of intersection of Γ with S′. If (3) is an integral of the first kind, then also $\int R(x, y)dx$ is of the same kind, so that in this case the theorem of Abel may be stated as follows:—

The sum of the values of the Abelian integral of the first kind

$$\int \alpha dX+\beta dY+\gamma dZ,$$

attached to an algebraic twisted curve Γ, *taken from a fixed origin to the points of intersection of this curve with a variable algebraic surface remains constant, if the coefficients of the equation of this surface vary in an arbitrary manner.*

2. Suppose now that the curve Γ is also represented by the intersection of the algebraic surfaces

$$\begin{aligned} f_1(X, Y, Z)&=0,\\ f_2(X, Y, Z)&=0, \end{aligned} \tag{6}$$

of degree m. Then

$$\begin{aligned} \frac{\delta f_1}{\delta X}dX+\frac{\delta f_1}{\delta Y}dY+\frac{\delta f_1}{\delta Z}dZ&=0,\\ \frac{\delta f_2}{\delta X}dX+\frac{\delta f_2}{\delta Y}dX+\frac{\delta f_2}{\delta Z}dZ&=0. \end{aligned} \tag{7}$$

From this

$$dY=\frac{-\frac{\delta f_1}{\delta X}\cdot\frac{\delta f_2}{\delta Z}+\frac{\delta f_2}{\delta X}\cdot\frac{\delta f_1}{\delta Z}}{\frac{\delta f_1}{\delta Y}\cdot\frac{\delta f_2}{\delta Z}-\frac{\delta f_2}{\delta Y}\cdot\frac{\delta f_1}{\delta Z}},$$

(1) As before δ stands here for the partial differentiation sign.

$$dZ=\frac{\frac{\delta f_1}{\delta X}\cdot\frac{\delta f_2}{\delta Y}-\frac{\delta f_2}{\delta Y}\cdot\frac{\delta f_1}{\delta Y}}{\frac{\delta f_1}{\delta Y}\cdot\frac{\delta f_2}{\delta Z}-\frac{\delta f_2}{\delta Y}\cdot\frac{\delta f_1}{\delta Z}}. \tag{8}$$

Putting now in integral (3)

$$\alpha=\frac{\frac{1}{3}}{\frac{\delta f_1}{\delta Y}\cdot\frac{\delta f_2}{\delta Z}-\frac{\delta f_2}{\delta Y}\cdot\frac{\delta f_1}{\delta Z}},\quad \beta=\frac{\frac{1}{3}}{\frac{\delta f_1}{\delta Z}\cdot\frac{\delta f_2}{\delta X}-\frac{\delta f_2}{\delta Z}\cdot\frac{\delta f_1}{\delta X}},\quad \gamma=\frac{\frac{1}{3}}{\frac{\delta f_1}{\delta X}\cdot\frac{\delta f_2}{\delta Y}-\frac{\delta f_2}{\delta X}\cdot\frac{\delta f_1}{\delta Y}},$$

which are rational functions of X, Y, Z, and as such admissible as values of α, β, γ, (3) reduces to

$$\int\frac{dX}{\frac{\delta f_1}{\delta Y}\cdot\frac{\delta f_2}{\delta Z}-\frac{\delta f_2}{\delta Y}\cdot\frac{\delta f_1}{\delta Z}}.$$

The analytic form of Abel's theorem is in this case

$$\sum_1^\mu\int_{x_0}^{x_i}\frac{dx}{\frac{\delta f_1}{\delta Y}\cdot\frac{\delta f_2}{\delta Z}-\frac{\delta f_2}{\delta Y}\cdot\frac{\delta f_1}{\delta Z}}=\text{const.}, \tag{9}$$

or also

$$\sum_1^\mu\frac{dx_i}{\frac{\delta f_1}{\delta Y_i}\cdot\frac{\delta f_2}{\delta Z_i}-\frac{\delta f_2}{\delta Y_i}\cdot\frac{\delta f_1}{\delta Z_i}}=0, \tag{10}$$

where $x_1, x_2, x_3, \ldots, x_\mu$ are the abscissas of the points of intersection of Γ with the variable surface.

§7. Application to Quartic in Space of the First Kind.

1. The parametric equations

$$x=a\,\mathrm{sn}u,\ y=b\,\mathrm{cn}u,\ z=c\,\mathrm{dn}u, \tag{11}$$

represent a curve of this kind, since they satisfy the equations

$$f_1 \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$
$$f_2 \equiv \frac{k^2x^2}{a^2}+\frac{z^2}{c^2}-1=0. \tag{12}$$

These represent two elliptical cylinders intersecting each other in the given quartic.

A plane

$$Ax+By+Cz+D=0 \tag{13}$$

intersects this curve in four points (x_i, y_i, z_i) $i=1, 2, 3, 4$. The application of Abel's theorem, (10), to (12) and (13) gives

$$\frac{dx_1}{y_1z_1}+\frac{dx_2}{y_2z_2}+\frac{dx_3}{y_3z_3}+\frac{dx_4}{y_4z_4}=0,$$

or, after replacing x, y, z by their values extracted from (11),

$$du_1+du_2+du_3+du_4=0,$$
$$u_1+u_2+u_3+u_4=\text{const.}, \tag{14}$$

where the u's are the arguments in (11) corresponding to the points of intersection of the plane with the quartic. To determine the constant in (14), let (13) coincide with the yz-plane. Then $x=a\,\mathrm{sn}u=0$ and from the theory of elliptic functions it is known that the sum of the arguments for which $\mathrm{sn}u=0$ is $\equiv 0$(mod. per.), or

$$u_1+u_2+u_3+u_4\equiv 0. \tag{15}$$

As every quartic in space may be transformed into the curve (12) *we have the theorem:—*

If a quartic in space of the first kind is parametrically represented by elliptic functions of the argument u, then the sum of the arguments of its points of intersection with any plane is congruent zero, modulus periodicity.

§8. Problems of Closure.

1. On a quartic Γ assume any two points P and v_1 and pass a plane through P and v_1 cutting Γ in two other points u_1 and u_2, then

$$P+v_1+u_1+u_2\equiv 0.$$

If P and v_1 remain fixed then $P+v_1\equiv c_1$ and also $-(u_1+u_2)\equiv c_1$ (constants). If the plane turns about Pv_1 then the ray joining u_1 and u_2 describes a hyperboloid of one sheet passing through Γ. Keeping P fixed and assuming another point v_2 on Γ so that $P+v_2\equiv c_2$ and passing a plane through P, v_2 and u_2, this plane will cut Γ in a fourth point u_3 so that $-(u_2+u_3)\equiv c_2$. The ray $\overline{u_2u_3}$ is on a second hyperboloid through Γ. Assume a third point v_3 on Γ, so that $P+v_3\equiv c_3$; pass a plane through P, v_3 and u_3, cutting Γ in a fourth point u_4, so that $-(u_3+u_4)\equiv c_3$. The ray $\overline{u_3u_4}$ lies on a third hyperboloid. Continuing this construction up to a point v_{2n} on Γ, giving finally a point u_{2n+1} on Γ, the table results:—

$$\begin{aligned}
&P+v_1+u_1+u_2\equiv 0,\\
1.\quad &-(u_1+u_2)\equiv c_1,\\
2.\quad &-(u_2+u_3)\equiv c_2,\\
3.\quad &-(u_3+u_4)\equiv c_3,\\
&\cdot\\
&\cdot\\
&\cdot\\
n-1.\quad &-(u_{2n-1}+u_{2n})\equiv c_{2n-1},\\
n.\quad &-(u_{2n}+u_{2n+1})\equiv c_{2n}.
\end{aligned}\tag{16}$$

The number $2n$ has been chosen for exactly the same reason as in §3, 2. Subtracting the sum of the congruences of odd order from the sum of those of even order, the congruence results

$$u_1-u_{2n+1}\equiv \sum_1^n c_{2k}-\sum_0^{n-1} c_{2k+1}.\tag{17}$$

The last point v_{2k} or any other fixed point may always be uniquely chosen in such a manner that the right-hand member of (17) *vanishes.* Indeed,

$$\begin{aligned}
&P+v_1+u_1+u_2\equiv 0,\\
&P+v_2+u_2+u_3\equiv 0,\\
&\cdot\\
&\cdot\\
&\cdot\\
&P+v_{2n}+u_{2n}+u_{2n+1}\equiv 0,
\end{aligned}\tag{18}$$

from which

$$\sum_{0}^{n-1} v_{2n+1} - \sum_{1}^{n} v_{2n} + u_1 - u_{2n+1} \equiv 0, \tag{19}$$

hence, from (17)

$$\sum_{0}^{n-1} v_{2n+1} - \sum_{1}^{n} v_{2n} \equiv \sum_{0}^{n-1} c_{2k+1} - \sum_{1}^{n} c_{2n}, \tag{20}$$

which proves the proposition. Hence the theorem:—

The vertices of a closed polygon of 2n sides may be moved in such a manner on the base-curve of a pencil of quadrics (hyperboloids of one sheet) that each of its sides describes an hyperboloid of this pencil; 2n—1 of these hyperboloids may be chosen at random, and by these the last one is perfectly determined.

2. Assume a single hyperboloid passing through Γ and any generatrix of one of the ruled systems of the hyperboloid defined by $u_1+u_2\equiv c$; through u_2 pass a generatrix $\overline{u_2u_3}$ of the other system so that $-(u_2+u_3)\equiv c$; through u_3 pass again a generatrix $\overline{u_3u_4}$ of the first system; and so forth. To find the condition that such a polygon of generatrices closes we have, as before, an even number, and the congruences:—

$$\begin{aligned} u_1+u_2 &\equiv c, \\ -(u_2+u_3) &\equiv c, \\ u_3+u_4 &\equiv c, \\ &\cdot \\ &\cdot \\ &\cdot \\ u_{2n-2}+u_{2n}-1 &\equiv c, \\ -(u_{2n}+u_{2n+1}) &\equiv c. \end{aligned} \tag{21}$$

By addition

$$u_1 - u_{2n+1} \equiv 2nc.$$

In order that $u_1-u_{2n}+1\equiv 0$ we have

$$2nc = 4mk + 4im_1k_1$$

$$c = \frac{2mk + 2im_1k_1}{n}. \tag{22}$$

Thus, if u_1 is given,

$$u_2 \equiv \frac{2mk + 2im_1k_1 - nu_1}{n} \tag{23}$$

and

$$x_2 = \mathrm{sn}\left(\frac{2mk + 2im_1k_1 - nu_1}{n}\right). \tag{24}$$

This equation admits of n^2 solution if n is prime, and includes x_1.

If c and u_2 are determined according to (22) and (23), the polygon on the corresponding hyperboloid always closes. We have, therefore, the theorem:—

If n is prime, then there are $n^2 - 1$ hyperboloids passsing through the curve Γ, for each of which the polygons of $2n$ sides formed by its generatrices and inscribed to the curve are always closed.

3. Steiner's theorem may easily be derived from this theorem. Taking a point P on the curve Γ as a center of projection, and through P the two generatrices PA and PB on one of the hyperboloids H, determined by (22), and projecting Γ and all of the closed polygons upon H then the points A and B are projected into two fixed points v_1 and v_2 of the projected Γ, which is a plane cubic C_3. In the plane a closed polygon of $2n$ sides is obtained whose sides alternately pass through v_1 and v_2.

Similarly Steiner's generalized theorem results from the first theorem of this section by making a central projection with P as a center[1].

§9. Projective Theory of Problems of Closure on the Cubic.

1. Comparing geometrical and analytical methods by which problems of closure may be solved it is apparent that, analytically, theorems of closure may be proved with great simplicity and elegance. They all appear as special applications of Abel's great theorem, or more specifically as applications of elliptic functions.

The difficulty, however, arises when an attempt is made to actually exhibit the results obtained by this method. It can be done

(1) Kötter, loc. cit.

by plotting curves and using elliptic tables. This exceedingly tedious work is mechanical and without geometrical interest. The problem of actually and ultimately exhibiting the results of mathematical investigations, either graphically or in some other way accessible to technical purposes, is necessary. The most beautiful and effective method of representing problems of closure graphically is obtained by a combination of projective and descriptive geometry as it has been established principally by Fiedler[1] and Disteli, loc. cit. For the reasons mentioned above I shall give a short sketch of this method.

2. Two cones of the second order generally intersect each other in a curve of the fourth order of the first kind, C_4. Without loss of generality we may assume that each of these cones has a circular base in the plane of projection (plane of the paper) of a central projection. Designate these circles by L_1 and L_2 and the vertices of the cones by M_1 and M_2, Fig. 4. Join M_1 and M_2 by a straight line and find its trace S_{12} in the plane of projection. To simplify the designation I shall mark the projections with the same letters as their corresponding elements in space. Every auxiliary plane through M_1 and M_2 intersects each cone in two generatrices, and the four generatrices thus obtained intersect each other in four points of the C_4. The trace h of every auxiliary plane passes through S_{12} and cuts L_1 and L_2 each in two points which when connected with M_1 and M_2 furnish four generatrices in the auxiliary plane. If we now choose the centre of projection C on the curve C_4 itself then its projection will be a circular curve of the third order, C_3, since it passes through the four points of intersection of L_1 and L_2 of which two are the circular points at infinity. The vertices M_1 and M_2 are projected on L_1 and L_2 and the C_3 touches L_1 and L_2 at M_1 and M_2. In analogy with the theory of conics, every ray through C cutting the polar plane of C with respect to either cone in a point C′, cuts the same cone in two points X and Y so that $(C'PXY) = -1$; C′, P, X, Y form a harmonic group. To find the polar-planes Γ' and Γ'' of C with respect to the cones, connect C with M_1 and M_2 and produce these connecting-

[1] Darstellende Geometrie, Vol. II, pp; 148-182, Vol. III, pp. 320-363.

lines to their traces; from these draw tangents to L_1 and L_2 respectively; the lines joining the points of contact on each circle are the traces of Γ' and Γ''. It is evident that in the arrangement of Fig. 4 these traces are the tangents to L_1 and L_2 at M_1 and M_2. To find the tangent to the curve C_4 at any of its points P, find the traces of PM_1 and PM_2 and in these draw the tangents to L_1 and L_2; where these intersect is the trace of the required tangent. The tangent to C_3 at the projection of P also passes through this trace. If we now consider the tangent to the C_4 at C, it is in the first place clear that its trace is a point of C_3; secondly the tangents at M_1 and M_2 to L_1 and L_2 pass through this point S_t°.

Suppose now that in a general case the polar-planes of C, Γ' and Γ'' intersect each other in the line d. Pass a plane through C and d, intersecting the C_4 in four points D_1, D_2, D_3, D_4, and connect one of these, say D_1, with C and produce to the intersection D_1' with d; let R and Q be the remaining points of intersection of CD_1 with the cones, then, since d is common to both polar-planes it follows $(CD_1D_1'R) = -1$, $(CD_1D_1'Q) = -1$, i. e., R must coincide with Q, or CD_1 produced cuts the curve C_4 in another point, say D_2. Similarly, CD_3 produced cuts the C_4 in D_4. Hence the theorem:—

The polar-planes of the centre of projection with respect to two cones of the second order intersect each other in a straight line whose projection is the line containing the double-points of the projected curve.

In our case, Fig 4, this line is the tangent to the C_4 at C and its projection is the point S_t°.

The trace h through S_{12} of an auxiliary plane through M_1M_2 gives the four points ABCD (this C is, of course, different from the above C) as points of intersection of the four generatrices, two through each, M_1 and M_2.

ABCD is therefore a quadrilateral inscribed to the curve of the third order and with its sides alternately passing through M_1 and M_2, two fixed points of the curve C_3. The same is true of all quadrilaterals arising from choosing successively all possible traces through S_{12}. Thus, Steiner's theorem of closure on the cubic in the

case of a quadrilateral is obtained in a very simple manner by this descriptive-projective method.

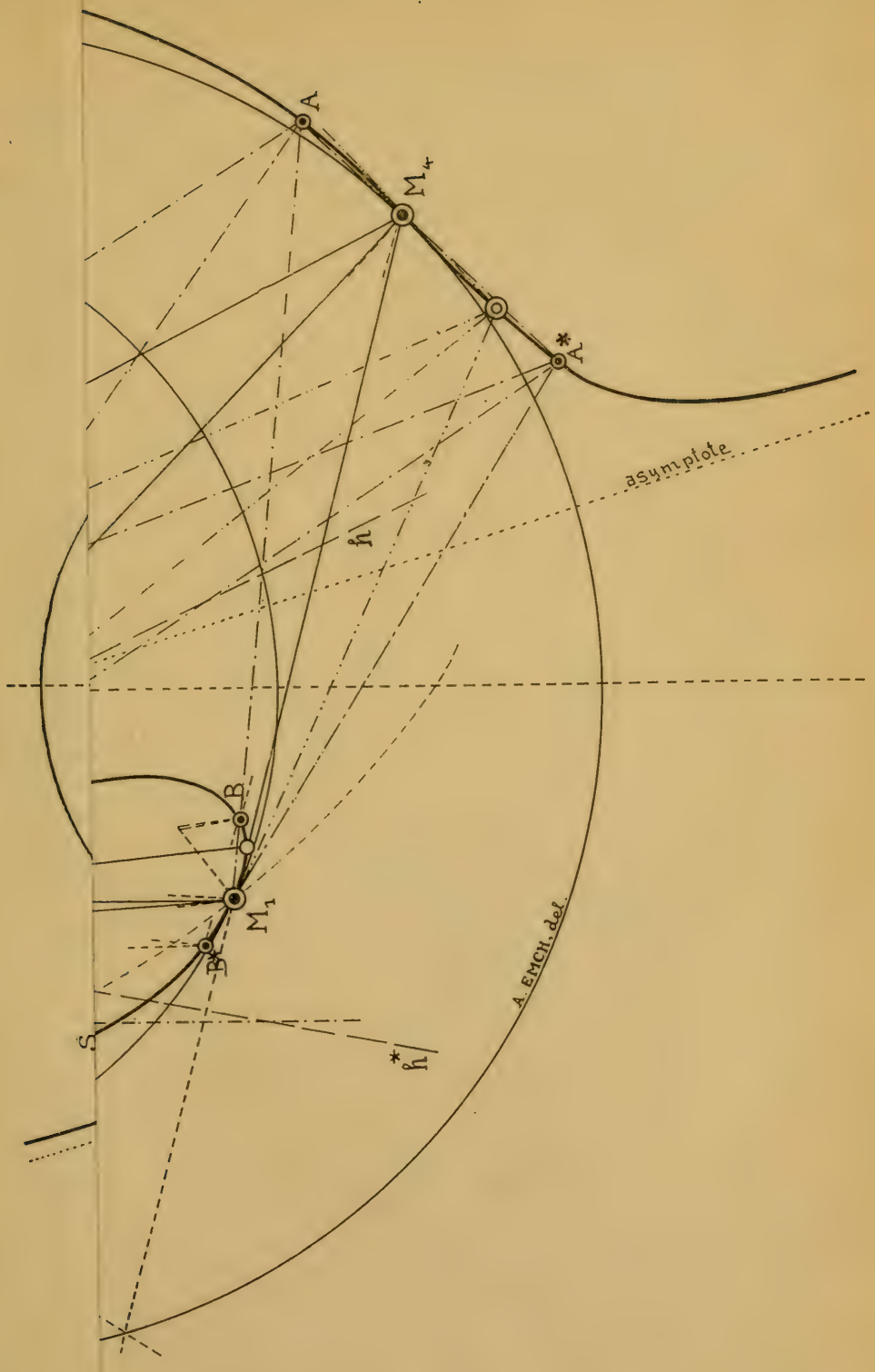
But this method also yields the theorem of Clebsch proved in §4.

3. The polar-lines of the pencil of planes through M_1M_2 with respect to the two given cones lie respectively in the polar-planes P_1, P_2 (traces p_1 and p_2) of M_1M_2 with respect to the same cones. Let P_1 and P_2 intersect each other in the line g. The pencil of planes through M_1M_2 and its corresponding pencil of polar-lines in P_1 cut g in an involution of points I_1. The same pencil of planes and its corresponding pencil of polar-lines in P_2 likewise cut g in an involution I_2. If we now construct the double-elements M_3 and M_4 of I_1 and I_2, then M_3 and M_4 have each the same polar-plane with respect to both of the given cones. Hence, if P is any other point of intersection of the given cones, i. e., a point of the C_4, and CP produced cuts the polar-plane of C in T and the given cones in F and G, we have again $(CPTF) = -1$, $(CPTG) = -1$, i. e., F and G must coincide. In other words, *any ray through M_3 and a point of the C_4 intersects the same curve in another point. The same is true of M_4.*

Both M_3 and M_4 are therefore vertices of cones of the second order containing the C_4 completely, and what has been said with regard to the first two cones also holds for these, and finally for any two among the four.

In the projection, M_3 and M_4 lie on the traces L_3 and L_4 of the last two cones, and the C_3 is also tangent to L_3 and L_4 at M_3 and M_4. The tangents at M_3 and M_4 also pass through S_t^c. In conclusion the theorem holds: *The points M_1, M_2, M_3, M_4 being the projections of the vertices of the four cones passing through the C_4 form a quadruple.*

To construct the traces of the other two cones, for instance of L_4, find first the trace S_{24} of M_4 M_2, which is obtained as the intersection-point of the traces p_2 and p_4 of the polar planes of M_2 and M_4 with respect to the corresponding cones. S_{24} is therefore the point of intersection of p_2 and M_2M_4. Through S_{24} draw a trace joining S_{24} with the point of intersection of M_2M_3 and L_2; this trace intersects M_4M_1 produced in a point of L_4. In a similar manner another point


A
M_4
A^*
asymptote
h
B
M_1
S
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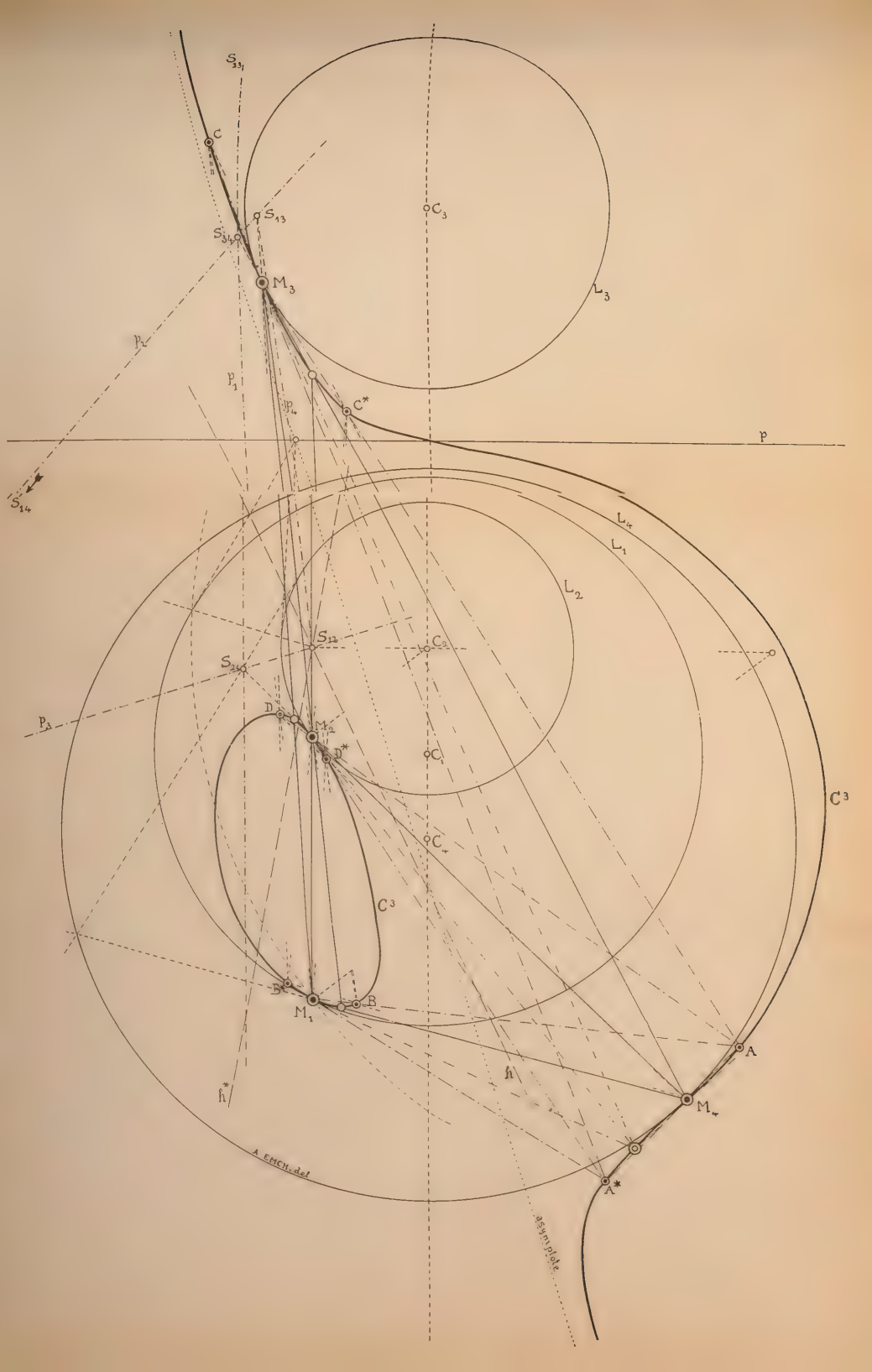
A. EMCH. del.


S_{31}
C
S_{13}
S_{34}
C_3
M_3
L_3
p_2
p_1
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C^*
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S_{14}
L_4
L_1
L_2
S_{12}
C_2
S_{24}
p_3
D
M_2
D^*
C_1
C_4
C^3
C^3
B
B^*
M_1
h
h^*
A
M_4
A^*
asymptote

of L_4 and thus L_4 itself may be determined. L_3 may be found in a similar manner.

Through the C_4 a pencil of quadrics may be passed of which the four cones are the singular surfaces. The traces of all these surfaces form a pencil of conics, and as four of them are circles (L_1, L_2, L_3, L_4), the whole pencil consists of a system of coaxial circles with the radical axis p. The tetrahedron $M_1M_2M_3M_4$ is self-polar in the pencil of quadrics through C_4. Any ray through the vertex, for instance M_1, containing two points A and B of the C_4, cuts the opposite polar-plane $M_2M_3M_4$ in a point P so that $(M_1PAB) = -1$. The tangents to the C_4 at A and B meet in a point of the polar-plane.

4. Through M_1M_2 pass two planes H and H* forming a harmonic pencil with the planes $M_1M_2M_3$ and $M_1M_2M_4$. The traces h, h^*, p_3, p_4 of these planes form a harmonic pencil. Each of these auxiliary planes cuts the curve in four points A, B, C, D and A*, B*, C*, D*, and as these are all in harmonic planes with respect to the foregoing polar-planes it follows that, two by two, they are also situated on rays through M_3 and M_4. Thus the rays AA*, BB*, CC*, DD* pass through M_4 and AC*, BD*, CA*, DB* through M_3. On account of the harmonic division the double secants BD and B*D* of the C_4 meet in a point of M_1M_2; similarly AC and A*C* meet in a point of M_1M_2. These two points are harmonic with M_1 and M_2. By analogy similar relations are found on the other edges of the self-polar tetrahedron. As A may be any point it follows that *every secant of the C_4 intersecting two opposite edges of the tetrahedron, intersects the curve in another point.*

As there are three pairs of opposite edges, the systems of double secants of the C_4 intersecting these pairs form three ruled surfaces and these are of the 4th order and 4th class.

In conclusion the theorem may be stated:—

The eight points of the C_4 in two planes harmonic with respect to any two planes of the tetrahedron lie four times in groups of four on rays through the vertices and six times in groups of eight on two planes through the edges of the tetrahedron. Each tangent at one of the points of the group is met by four other tangents in points which are the intersections of the given tangent with the

planes of the tetrahedron. The diagonals of the six plane quadrilaterals formed by the eight points form three closed oblique quadrilaterals whose vertices are on the edges of the tetrahedron, and through each of these quadrilaterals passes a hyperboloid belonging to the pencil through the C_4.

Considering the projection of the whole configuration, Fig. 4 shows immediately that BCDA* and B*C*D*A are quadruples on the C_3, since they are the intersections of the pencils A. $M_1M_2M_3M_4$ and A.*$M_1M_2M_3M_4$ with the C_3. This proves Clebsch's first theorem. The second theorem appears from the fact that the sixteen lines joining the points of the quadruples BCDA* and B*C*D*A intersect each other in the quadruple $M_1M_2M_3M_4$. In the figure the point of intersection K of M_2M_3 and M_1M_4 is the trace of a ray through the center of projection C and cutting M_2M_3 and M_1M_4; this ray meets the C_4 in another point and its trace, being the projection of this point, necessarily belongs to the C_3. The same is true of the remaining diagonal points of the quadruple $M_1M_2M_3M_4$.

5. In preparing Fig. 4 I have shown only the most important parts of the construction. For a detailed descriptive-projective execution I refer to Fiedler, loc. cit.

It goes without saying that by the same method, employing higher involutions, closed polygons of $2n$ sides may be obtained. For the purpose of this paper it is sufficient to establish by this method some of the propositions of §8.

The complete geometrical theory of Steiner's polygons is given in Disteli's work, loc. cit.

III. APPLICATION OF ELLIPTIC FUNCTIONS TO CERTAIN LINKAGES.

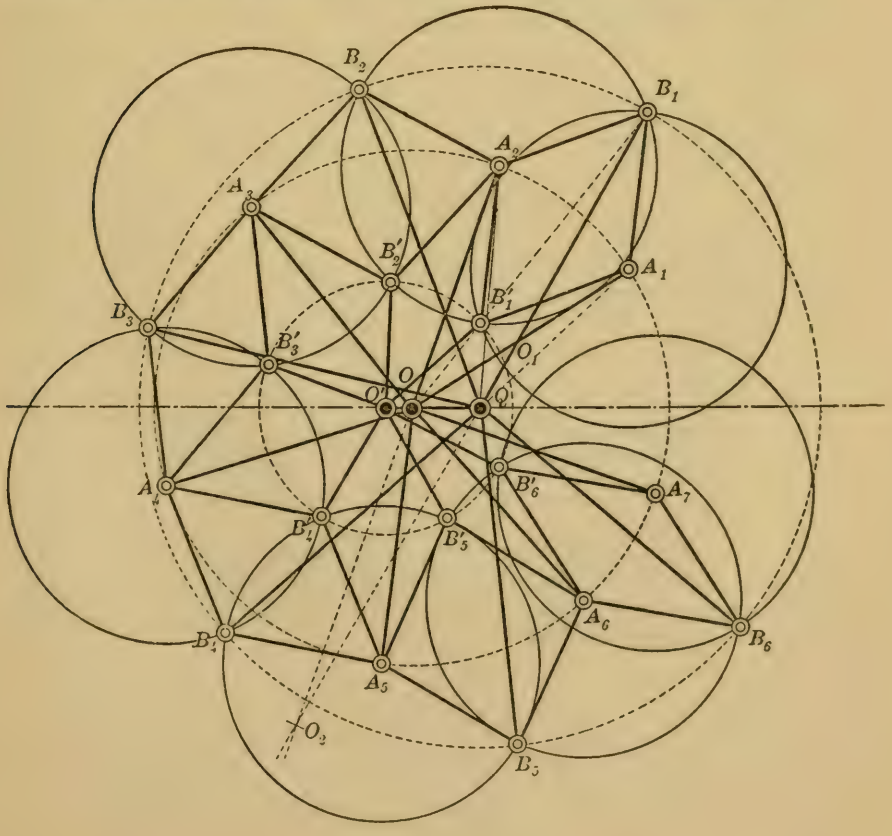
§10. Peaucellier's Inversor.

1. The element of the linkage which I shall consider consists of a *Peaucellier's Cell or Inversor*[1].

(1) A description of this particular link-work may be found in every modern text-book on kinematics. The constructions that it performs were proposed by Peaucellier in the *Nouvelles Annales de Mathématiques*, ser. 2, vol. 3 (1864), p. 414, and the link-work was published by him in the same journal, ser. 2, vol. 12 (1873), p. 71.

An account of the application of elliptic functions to Peaucellier's inversor appeared in the Annals of Mathematics, 2nd series, Vol. 2, No. 2.

Assume in Fig. 5, the links $OA_1=OA_2=r_1$, $A_1B_1=B_1A_2=A_2B_1'=B_1'A_1=r_2$, $QB_1=R$, and $OQ=e$. The points O and Q are fixed, while all the others are movable. During the motion B_1 describes a circle having Q as a center and QB_1 as a radius. Now, according to the properties of Peaucellier's Inversor $OB_1.OB_1'=r_1^2-r_2^2=$constant; consequently the point B_1' also describes a circle, which is inverse to the circle described by B_1 in an inversion having O as a centre and $\sqrt{r_1-r_2^2}$ as a radius. Further let δa_1, δa_2, $\delta\beta_1$ be the infinitesimal displacements of the points A_1, A_2, B_1 in a virtual displacement of the cell; a_1, a_2 the angles which the links OA_1, OA_2 include with the positive part of the axis OQ; β_1, β_2 the angles which the links A_1B_1 and B_1A_2 include with the link QB_1; O_1 and O_2 the points of intersection of the link QB_1 with the links OA_1 and OA_2 respectively; and, finally, the variable distances $\rho_1=QA_1$ and $\rho_2=QA_2$. The points O_1 and O_2 are evidently the virtual centres of rotation of the links A_1B_1 and B_1A_2 respectively. Hence, from Fig. 1, the relations:—



$$\frac{\delta a_1}{\delta \beta_1} = \frac{A_1O_1}{B_1O_1}, \qquad (1) \qquad \frac{\delta a_2}{\delta \beta_2} = \frac{A_2O_2}{B_1O_2}, \qquad (2)$$

from which follows

$$\frac{\delta a_1}{\frac{A_1O_1}{B_1O_1}} = \frac{\delta a_2}{\frac{A_2O_2}{B_1O_2}}. \qquad (3)$$

Now $$\frac{A_1O_1}{B_1O_1} = \frac{\sin \beta_1}{\sin \gamma}, \qquad \frac{A_2O_2}{B_1O_2} = \frac{\sin \beta_2}{\sin \gamma},$$

where *angle* $OA_1B_1 = angle\ OA_2B_1 = \gamma$. Consequently,

$$\frac{\delta a_1}{\sin \beta_1} = \frac{\delta a_2}{\sin \beta_2}. \qquad (4)$$

As there is only one degree of freedom, the angles β_1, β_2, a_1, and a_2 may be regarded as functions of the same independent variable. From triangle OA_1Q follows

$$cos a_1 = \frac{r_1^2 + e^2 - \rho_1^2}{2r_1e},$$

and by differentiation

$$\sin a_1 \cdot da_1 = \frac{\rho_1 \cdot d\rho_1}{r_1e}.$$

But $$\sin a_1 = \frac{2}{r_1e}\sqrt{s(s-r_1)(s-e)(s-\rho_1)},$$

where $s = \frac{r_1+e+\rho_1}{2}$, or

$$\sin a_1 = \frac{\sqrt{-[\rho_1^2 - (r_1+e)^2]\,[\rho_1^2 - (r_1-e)^2]}}{2r_1e},$$

hence $$da_1 = \frac{2\rho_1 \cdot d\rho_1}{\sqrt{-[\rho_1^2 - (r_1+e)^2][\rho_1^2 - (r_1-e)^2]}}.$$

In the triangle $A_1B_1O_1$

$$\sin \beta_1 = \frac{\sqrt{-[\rho_1^2-(R+r_2)^2][\rho_1^2-(R-r_2)^2]}}{2Rr_2},$$

so that

$$\frac{d\alpha_1}{\sin \beta_1} = \frac{4Rr_2\rho_1 \cdot d\rho_1}{\sqrt{[\rho_1^2-(r_1+e)^2][\rho_1^2-(r_1-e)^2][\rho_1^2-(R+r_2)^2][\rho_1^2-(R-r_2)^2]}}.$$

To abbreviate let $(R+r)^2=a$, $(r+e)^2=b$, $(r-e)^2=c$, $(R-r)^2=d$, $\rho_1^2=x$, $\rho_1 \cdot d\rho_1=\frac{1}{2}\,dx$, so that finally

$$\frac{d\alpha_1}{\sin \beta_1} = \frac{2Rr_2 \cdot dx}{\sqrt{(x-a)\ (x-b)\ (x-c)\ (x-d)}}. \tag{5}$$

In a similar manner, if $QA_2=\rho_2$, and $\rho_2^2=y$,

$$\frac{d\alpha_2}{\sin \beta_2} = \frac{2Rr_2 \cdot dy}{\sqrt{(y-a)\ (y-b)\ (y-c)\ (y-d)}}. \tag{6}$$

Putting

$$\int_c^x \frac{dx}{\sqrt{(x-a)\ (x-b)\ (x-c)\ (x-d)}} = u, \tag{7}$$

$$\int_c^y \frac{dy}{\sqrt{(y-a)\ (y-b)\ (y-c)\ (y-d)}} = v, \tag{8}$$

according to equation (4) we have:—

$$v-u=h \text{ (constant)}. \tag{9}$$

By inversion of the elliptic integrals (7) and (8) the elliptic functions

$$x=\lambda(u), \qquad y=\lambda(v) \tag{10}$$

are obtained. In this manner the cosines of the angles α_1 and α_2 may be rationally expressed by elliptic functions, and it is found that *the difference of the arguments belonging to these angles is constant and independent of the position of the cell.*

2. As indicated in Fig. 5, other equal cells $(OA_2B_2A_3 \cdot B_2Q)$, $(OA_3B_3A_4 \cdot B_3Q)$ . . . may be added to the first, which together form a general link-work. In this process of adding cells two prin-

cipal cases may occur: (1) the link-work will close after a certain number of additions of cells, i. e., the last point A obtained in the construction will coincide with the first of the points A; (2) the link-work does not close.

To discuss the conditions of a closed link-work assume that there are n cells in it, so that the point A_{n+1} of the nth cell $OA_nB_nA_{n+1}$. QB_n will coincide with the first point A_1. The argument belonging to the angle a_1 or the point A_1 being u, the argument of A_2 will be $u+h$, of $A_3 u+2h$, . . . , of $A_{n+1}u+nh$. But A_{n+1} coincides with A_1, hence, designating the periods of the elliptic function $\lambda(u)$ by $2w$ and $2w$,

$$u+nh \equiv n \pmod{2w, 2w_1}.$$

This condition is satisfied if

$$h \equiv 0 \left(\mathrm{mod}\frac{w}{n}, \frac{w_1}{n}\right),$$

or

$$h = \frac{2m2w+2m_1 2w_1}{n}, \tag{11}$$

where m_1 and m_2 designates integers. Consequently the problem of a closed link-work is solved if h is given one of the values contained in (11). This condition necessarily requires a special arrangement of the link-work; but it does not assign any particular value to the argument u. Thus, the first point A_1 of the link-work may be chosen anywhere on the circle having O as a centre and OA_1 as a radius; the link-work closes every time and contains n cells. This result may be stated in the theorem:—

If a link-work of the prescribed kind, based upon two fixed circles (centres O and Q, radii OA_1 and QB_1) closes and contains n cells, then every other link-work based upon the same two circles closes and contains n cells.

It is clear that the fundamental relation (3) also holds in the cases of limited and unlimited link-works. The previous theorem, however, only holds for a closed link-work.

§11. Geometrical Transformation of the Link-Work.

In Fig. 5, with A_1, A_2, A_3, . . . as centres and r_2 as a radius

describe a series of circles. From the figure it is seen that the two circles having A_i and A_{i+1} as centres pass through the points B_i and B_i'. The properties of closing of these series are evidently the same as those of the link-work. All circles of the series are tangent to two concentric circles. The figure can be generalized by an inversion and we have immediately the result:—

If each pair of consecutive circles of a series of circles, which all touch two fixed circles C_1 and C_2, intersect in two points B_i and B_i', and if the points B_1, B_2, B_3, . . . are situated on a circle C_3, then the points B_1', B_2', B_3', . . . are also situated on a circle C_4[1] (Fig. 6).

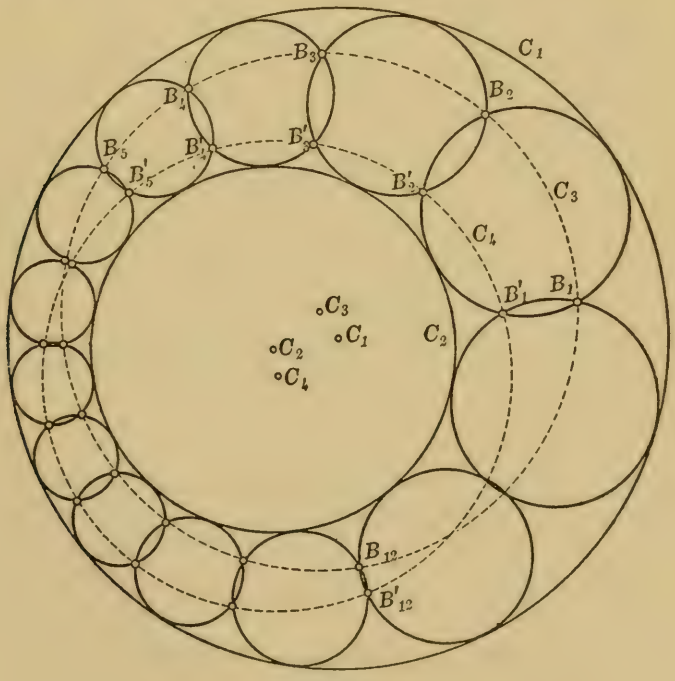


Fig. 6.

(1) This statement may be generalized in such a manner that instead of a circle C_3 any curve is assumed. From Fig. 1 it can easily be proved that in this case B_1', B_2', B_3', . . . are situated on a curve which is the inverse of the first with regard to the center O. The special case above has been formulated in view of its subsequent application.

As a geometrical transformation of the link-work does not affect the properties of closing of the above series of circles we may state, in analogy with the link-work:

If a series of circles (generalized Steinerian series) of the prescribed kind, based upon two fixed circles C_1 and C_2 and a third circle C_3 (or C_4), closes and contains n circles, then every other series of circles based in the same manner upon the same three circles closes and contains n circles.

By special disposition and by assuming some of the given circles as points or straight lines a great variety of circular series may be obtained. If the circles C_1, C_2 and C_3 are parts of a pencil of circles, then C_4 belongs to the same pencil. The series of circles obtained by this arrangement have been considered by Steiner.[1] One of the most interesting cases arises if one of the circles C_1 and C_2, for instance C_2, degenerates into point O. All circles of the series pass through O, and the circle C_4 coincides with O. Any inversion having O as a centre transforms all circles of the series into straight lines which are inscribed in a circle C_2' and circumscribed about a circle C_1' i. e., the limited portions of these lines form a polygon which is inscribed in one and circumscribed about another circle. The properties of closing of these polygons, which are called Poncelet's polygons, are the same as those of the general series of circles. Poncelet's constructions also result directly from a geometrical transformation of the link-work in which $r_1=r_2$.

§12. Special Cases.

1. I shall treat of the specializations referred to above in detail. Let $r_1=r_2=r$, then in order to reduce the integral

$$\int \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$$

to Legendre's normal form, we have to notice that in the case of an unlimited motion $2r>R+e$, or $(r-e)>(R-r)$, or $(r-e)^2>(R-r)^2$. But we have also $R+r>r+e$, and $r+e>r-e$, hence $R+r>r+e>r-e>R-r$, or

(1) *Werke*, Vol. I, pp. 19-76.

$$a>b>c>d. \tag{12}$$

In our case we always have $b>x>c$, so that according to a well-known formula[1]

$$\int_c^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \operatorname{sn}^{-1} \sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}}, \tag{13}$$

with the modulus $\kappa^2=k=\dfrac{(b-c)(a-d)}{(a-c)(b-d)}$.

Putting this integral, as in formula (5), equal to u, we have:—

$$\sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}} = \operatorname{sn}\left(\sqrt{\frac{(a-c)(b-d)}{2}}\cdot u\right), \tag{14}$$

or, putting $\sqrt{\dfrac{(a-c)(b-d)}{2}}\cdot u=z$,

$$\frac{(x-c)(b-d)}{(x-d)(b-c)}=\operatorname{sn}^2 z. \tag{15}$$

From this

$$x=\frac{c(b-d)-d\ (b-c)\ \operatorname{sn}^2 z}{(b-d)-(b-c)\ \operatorname{sn}^2 z}. \tag{16}$$

For $u=0$, $x=c=(r-e)^2$. The corresponding value of y is easily found as

$$y=re+\frac{r^3-e\mathrm{R}^2}{r-e}=p. \tag{17}$$

This value of y belongs to the argument $v=h$, since $v-u=h$; hence the constant h is determined by

$$re+\frac{r^3-e\mathrm{R}^2}{r-e} = \frac{c(b-d)-d(b-c)\ \operatorname{sn}^2\left(\sqrt{\frac{(a-c)\ (b-d)}{2}}\cdot a\right)}{b-d-(b-c)\ \operatorname{sn}^2\left(\sqrt{\frac{(a-c)\ (b-d)}{2}}\cdot a\right)}. \tag{18}$$

Designating the real half-period of sn z by 2K, we have:—

(1) See Greenhill, Elliptic Functions, pp. 53-55.

$$\text{sn}\ (z+2\text{K}) = -\text{sn}\ z,$$

or $$\text{sn}^2(z+2\text{K}) = \text{sn}^2 z,$$

i. e., 2K is the real period of $\text{sn}^2 z$. For $z=0$, $\text{sn}^2 z=0$ and $x=(r-e)^2$. For $z=\text{K}$, $\text{sn}^2 z=1$ and $x=b=(r+e)^2$. For $z=2\text{K}$, $\text{sn}^2 z=0$ and $x=c=(r-e)^2$. To find the corresponding value of x, belonging to $z=\text{K}|2$, we make use of the formula[1]:—

$$\text{sn}\ \frac{\text{K}}{2} = \frac{1}{\sqrt{1+\sqrt{k'}}} \tag{19}$$

where

$$k' = \frac{(a-b)\,(c-d)}{(a-c)\,(b-d)}$$

is the complementary modulus. Thus, for $z=\text{K}|2$, from formula (16) we obtain

$$x = \frac{b(c-d)+c(b-d)\sqrt{k'}}{(c-d)+(b-d)\sqrt{k'}}. \tag{20}$$

2. *Example of 3 Cells.* As the period of $\text{sn}^2 z$ is 2K, we have to put $z=2\text{K}$, in order to obtain the relation of R, r, e, in this particular closed link-work. Designating sn $z|3$ simply by S, we have:

$$\text{sn}\ z = \frac{3\text{S}-4(1+k)\,\text{S}^3+6k\text{S}^5-k^2\text{S}^9}{1\ -6k\text{S}^4+4(1+k)k\text{S}^6-3k^2\text{S}^8}, \tag{21}$$

and since sn $2\text{K}=0$, the condition becomes

$$k^2\text{S}^8-6k\text{S}^4+4(1+k)\text{S}^2-3=0. \tag{22}$$

According to formulas (17) and (15):—

$$\text{S}^2 = \frac{(x-c)(b-d)}{(x-d)(b-c)}.$$

Designating this expression by q, the required condition is

$$k^2q^4-6kq^2+4(1+k)q-3=0. \tag{23}$$

(1) For the formulas used and developed here and in the next two sections we refer to Greenhill, loc. cit., pp. 120-121.

Substituting in this expression the values of k and q in terms of R, r, e, it is easily found that condition (23) reduces to

$$R=r. \tag{24}$$

Thus, the three cell link-work is completely determined by fixing r and c. In Fig. 7, $OQ=e$ and $OA_1=r$. Now $R=r$, hence, in this case, $A_1B_1=QB_1$. Having fixed the point B_1 it is an easy matter to complete the construction of

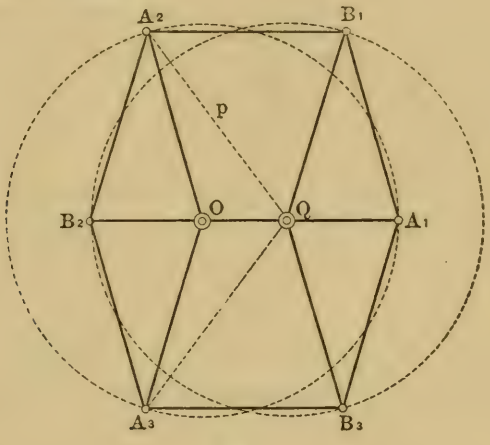


FIG. 7.

the cells $A_1B_1A_2O$, $OA_2B_2A_3$, $OA_3B_4A_1$. It is seen that $QB_1=QB_2=QB_4$, so that also $QB_2A_2B_1$, $QB_1A_1B_4$, $QB_4A_3B_2$ may be considered as cells of the link-work. The points O, A_1, A_2, A_3 may be interchanged with the points Q, B_1, B_2, B_3 without changing the character of the link-work.

3. *Example of 4 Cells.* In this case the value of y as given by formula (17) is also equal to the value of x in formula (20), i. e.,

$$p=\frac{b(c-d)+c(b-d)\sqrt{k'}}{(c-d)+(b-d)\sqrt{k'}}.$$

Substituting in this equation for a, b, c, d, p, and k' their values in terms of R, r, e, the conditions between R, r, and e is found:—

$$2r^2=R^2+e^2. \tag{25}$$

In this case the value of $x=p$ is

$$p=r^2-e^2=R-r^2,$$

as is also seen from Fig. 8, in which $x=\overline{QA_2}^2=\overline{OA_2}^2-\overline{OQ}^2=r^2-e^2$,

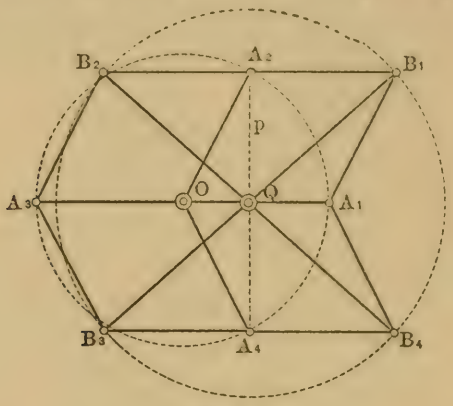


Fig 8.

and also $x=\overline{QB_2}^2-\overline{A_2B_2}^2=R^2-r^2$. From this figure it is apparent that during the motion the following groups of parallel links are maintained:—

$$A_1B_1 \parallel OA_2 \parallel A_3B_2, \qquad A_1B_4 \parallel OA_4 \parallel A_3B_3,$$
$$B_1A_2 \parallel OA_1 \parallel B_4A_4, \qquad B_2A_2 \parallel OA_3 \parallel B_3A_4.$$

It follows from this that during the motion

$$B_1B_4=A_2A_4=BB_3,$$

and

$$B_1B_2=A_1A_3=B_4B_3.$$

Consequently, the points $B_1B_2B_3B_4$ always form a parallelogram, in

which $QB_1=QB_2=QB_3=QB_4.$

But $B_1B_3=QB_1+QB_3,$

and $B_2B_4=QB_2+QB_4,$

hence $B_1B_3=B_2B_4.$

The parallelogram has, therefore, equal diagonals, and is a rectangle. The closed link-work is, consequently, also completely determined by

connecting the points B_1 and B_3, and B_2 and B_4 by links of equal length, and assuming

$$2r = \sqrt{2(R^2+e^2)} > R+e,$$

where e is any real quantity satisfying the implied condition.

These two links always cross each other at a point Q which does not change its distance from O during the motion.

4. *The Open Link-Work.* Consider a link-work of the prescribed kind which does not close or which is not completed so as to form a closed link-work. Suppose there are m cells in the link-work, and that the last cell does not overlap the first.[1] In this manner an angle $A_{m+1}OA_1$ is formed between the last and first cell. This angle, which will be designated by ϕ, is variable during the motion, and can be expressed by elliptic functions, for,

$$\phi = a_{m+1} - a_1 \tag{26}$$

is a function of the argument u.

The condition for a maximum or minimum of the angle ϕ is

$$\frac{d\phi}{du} = \frac{da_{m+1}}{dx_{m+1}} \cdot \frac{dx_{m+1}}{du} - \frac{da_1}{du_1} \cdot \frac{dx_1}{du} = 0. \tag{27}$$

According to previous formulas

$$\frac{da}{dx} = \frac{1}{\sqrt{-(x-b)(x-c)}}, \text{ and } \frac{dx}{du} = \sqrt{(x-a)(x-b)(x-c)(x-d)}.$$

Substituting these expressions, with the proper indices, in (27), the condition reduces to

$$(x_1-a)(x_1-d) = (x_{m+1}-a)(x_{m+1}-d),$$

or

$$x_1^2 - x_{m+1}^2 = (a+d)(x_1 - x_{m+1}). \tag{28}$$

This equation is satisfied in two ways:—

(1) when $x_1 = x_{m+1}$, (29)

(2) when $x_1 + x_{m+1} = a+d = 2(R^2+r^2)$. (30)

(1) This assumption is made in order to have a clearer idea of the link-work, although the results hold also in the most general case.

In the first case the condition $x_1 = x_{m+1}$ does not assign any relation between R, r, and e and holds therefore for every proper link-work.

Considering a complete revolution of a link-work, Fig. 1, it can easily be proved that there are only two positions of the link-work possible where $x_1 = x_{m+1}$. This is the case every time that the cell has a symmetrical position with regard to the axis OQ, which, in these cases, bisects the open space of the link-work. Suppose now that the link-work makes a complete revolution, starting from the position of the maximum angle. The angle cannot pass through zero, because the system would then be permanently closed, so that there must be a minimum between the two maxima. Similarly there must be a maximum between two minima. This result may be summed up in the theorem:—

The angle formed by an open link-work can assume only one maximum and one minimum during a complete revolution.

The maximum and minimum angles are both bisected by the diameter OQ.

If the angle becomes zero, it will remain zero. In this case we still have $x_1 = x_{m+1}$ (coincident); but for every position of the link-work. Thus, we see that the case of a closed link-work is included in case (1). The second condition $x_1 + x_{m+1} = 2(R^2 + r^2)$ can only be satisfied in a singular case, since $x_1 + x_{m+1}$, for all possible link-works, with constant values of R and r, may be considered as a function of m and e, having for all values of m and e a constant value. From formula (16) it appears that $x_1 + x_{m+1}$ can be independent of m and e only if $e = 0$. In this case $x_1 + x_{m+1} = 2r^2$, and, according to (30), $R = 0$. There is no proper link-work.

Without entering into mechanical details of the link-work it is interesting to mention the seemingly paradoxical fact, that all our link-works have one degree of freedom in their motion, although the closed link-work satisfies the condition of a rigid frame-work.

5. *Geometrical Transformation of the Link-Work.* With A_1, A_2, A_3, . . ., in the previous figures, as centres and r as a radius describe a series of circles as before. These circles all pass through O and intersect the circle of centre Q and radius R in the points B_1, B_x;

B_1, B_2; B_2, B_3; B_3, B_4; . . . respectively. In a closed link-work this series of circles closes also, so that the last point of intersection B_{n+1} will coincide with the first point B_x. This result may be stated in the following form:—

If two fixed circles A and B are given, a series of circles can be drawn, whose centres $A_1A_2A_3$, . . . all lie on the circle A and which all pass through the centre O of A. The first circle A_1 of this series intersects circle B in two points B_x. The second circle A_2 passes through B_1 and intersects circle B a second time in B_2. The third circle passes through B_2 and intersects B a second time in B_3, and so forth. In this manner a series of circles is obtained which may be divided into three different classes:—

I. The series is limited, i. e., the construction cannot be continued indefinitely.

II. The series closes, i. e., after the construction of a certain number of circles, the last point of intersection B_{n+1} will coincide with the first B_1.

III. The series is unlimited.

According to the general theorem on the link-work it follows immediately that if the series of circles closes once, it will close in all cases, no matter where the first circle of the series is drawn. If the series does not close in one case, it never will close.

The circles of the previous series all touch a circle C of centre O and radius $2r$. Applying to this series an inversion with centre O and any radius, every circle of the series is transformed into a straight line segment, tangent to the transformed circle of C and inscribed to the transformed circle of A. *Thus the series becomes a polygon which is inscribed to one and circumscribed to the other circle.* This is precisely the case of *Poncelet's polygons*,[1] Fig. 9.[2] As to the properties of closing of these polygons, it is evident that they are the same as in our link-work and the series of circles derived from it. The system of circles from which Poncelet's polygons arise may also

(1) In Poncelet's *Traite des proprietés projectives des figures* (1822) §565. See also Greenhill, *Elliptic Functions*, pp. 121-130.

(2) In Fig. 9, C has been chosen as circle of inversion.

be considered as a special case of *Steiner's circular series*, which, in general, consists of all circles tangent to two fixed circles. From these circles a *special* series may be selected in which one point of intersection of each pair of consecutive circles always lies on a third fixed circle. These series also include the cases of Steiner's circular series where each pair of consecutive circles intersect each other under a a constant angle. If this angle is zero two consecutive circles are always tangent to each other. If the first of the fixed circles of Steiner's special circular series contracts into the centre of the second fixed circle, the series arises from which Poncelet's polygons were obtained by an inversion as illustrated in Fig. 9.

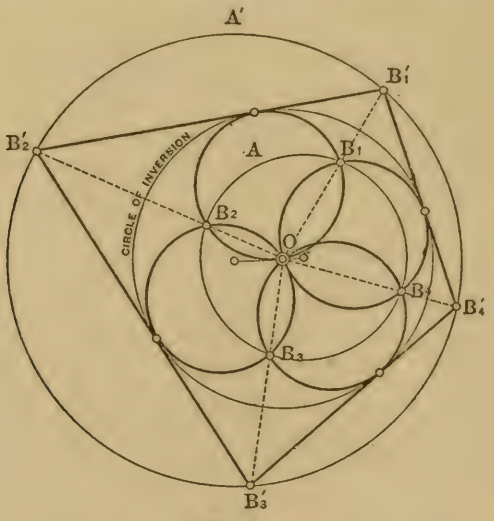


FIG. 9.

§13. CIRCULAR TRANSFORMATIONS, CONJUGATE SERIES.

1. Representing the above configurations in a complex plane it is known that a circular transformation

$$z' = \frac{az+b}{cz+d}$$

does not change properties of closure and conformity (isogonality

and tangency). The properties of Steiner's circular series are therefore not changed by such a transformation.

From this standpoint it is an easy matter to prove the following theorem: *If in a coaxial system of circles* $A_1, A_2, A_3, \ldots, A_k, \ldots$, *with real limiting points, a system of circles* $B_1, B_2, B_3, \ldots, B_n$ *can be described each of which being tangent to any two circles A, and so that* B_1 *cuts a fixed circle* A_k *of the coaxial system in* P_1 *and* P_2, B_2 *cuts* A_k *in* P_2 *and* P_3, B_3 *in* P_3 *and* $P_4, \ldots,$ *finally* B_n *cuts* A_k *in* P_n *and* P_1, *then, under the same conditions, an infinite number of closed systems of circles B can be described. In all these closed systems the points of intersection of corresponding consecutive circles lie in circles of the same coaxial system.*

In reality, an inversion having one real limiting point as the centre of inversion transforms the system (A) into a system of concentric circles, and in connection with this the theorem is evident.

If the circles B all pass through one real limiting point, being thus tangent only once to circles of (A), and making an inversion, Poncelet's theorem of the polygon inscribed to a circle and tangent to circles of a concentric system is obtained. By considerations similar to those in connection with linkages the theorem is deduced:—

If a system of circles B pass through any fixed point Q and if each circle B is tangent to a circle of a coaxial system (A) *and cuts the preceding and consecutive circle of* (B) *in points of a fixed circle of* (A), *then there are an infinite number of such closed systems* (B).

From an inversion with Q as a centre results Poncelet's general theorem concerning closed polygons inscribed to a fixed circle and tangent to circles of a coaxial systems.

2. The algebraic properties of all these configurations have been studied by A. Hurwitz,[1] who has shown that *they rest upon the existence of more than n roots of an equation of degree n.*

Cayley in a number of articles[2] reduces all problems of this kind to the differential equation $\frac{dx}{\sqrt{f(x)}} = \frac{dy}{\sqrt{f(y)}}$, f being a polynomial

(1) Mathematische Annalen, Vol. 15, pp. 8-15, and Vol. 19, pp. 56-66.

(2) See in particular, "On the Porism of the In- and Circumscribed Polygon," etc., Cayley's collected Mathematical Papers, Vol. VIII, pp. 14-21.

of the fourth degree. This is also our idea for the solution of closed linkages.

The properties of circular series may be extended to space. In this connection I state without proof the theorem[1]:—

Any closed series of tangent-spheres formed by generating spheres of one system of a Dupinian cyclide is conjugate to any closed series of the other system.

If the ratio of the number of revolutions into the number of spheres of one closed series is $\frac{k_1}{m_1}$, *then every series of the other system closes, and the ratio* $\frac{k_2}{m_2}$ *belonging to these series is such that*

$$\frac{k_1}{m_1}+\frac{k_2}{m_2}=\frac{1}{2}.$$

The same property holds for circular series and their conjugates in a plane, as Steiner has stated in a celebrated proposition[2].

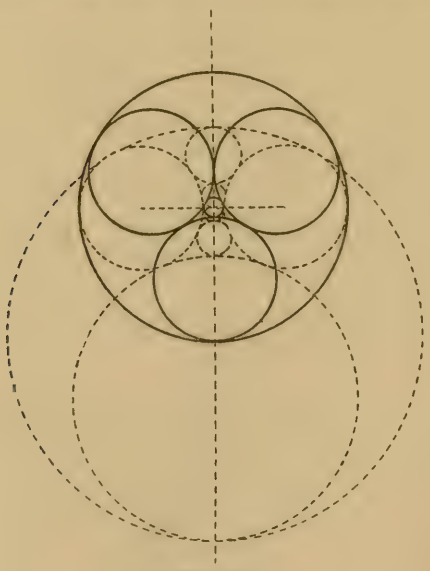


FIG. 10.

(1) See Annals of Mathematics, Vol. XII, p. 159.
(2) Loc. cit., p. 43.

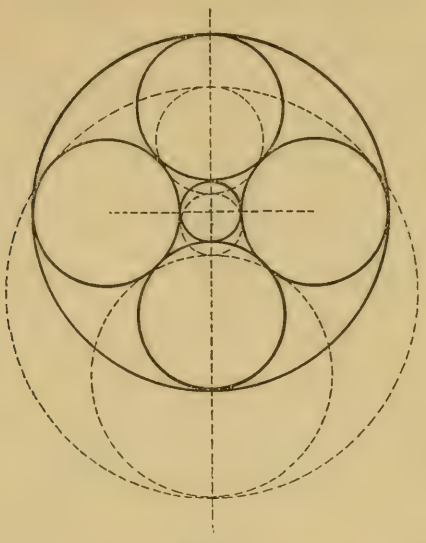


FIG 11.

As an example take $k_1 = 1$, $m_1 = 3$, then $\frac{k_2}{m_2} = \frac{1}{6}$, hence a conjugate series has one revolution and six circles. In Figs. 10, 11, the circles of the original series are drawn in solid lines, while those of the conjugate series are represented by dotted lines.

To find two conjugate series, each having the same number of revolutions and circles, put $\frac{k_1}{m_1} = \frac{k_2}{m_2}$, which requires each of these ratios to be $\frac{1}{4}$. This case is illustrated in Fig. 11.

IV. LOXODROMICS ON THE TORUS.

1. Writers on the theory of surfaces frequently point to the torus as an instructive example. DR. REPETTO in a monograph "Sulle Geodetiche del Toro,"[1] has made a thorough investigation of the geodesics of the torus and states the condition for closed curves

(1) University of Sassari, Italy. See also article by Dr. G. A. Bliss, in Annals of Mathematics, Vol. IV, No. 1.

of this kind. In the present chapter I shall discuss the loxodromics of the torus and their properties of closure.[1]

2. Designating by u and v the angles, which, in Fig. 12, determine the position of a point B on the surface of a torus, the square of a linear element on the surface has the form,

$$ds^2 = (\mathrm{R}+r\sin v)^2(du_1^2+dv_1^2), \tag{1}$$

$$\text{where } u_1 = u \text{ and } v_1 = \int_0^v \frac{dv}{\frac{\mathrm{R}}{r}+\sin v}. \tag{2}$$

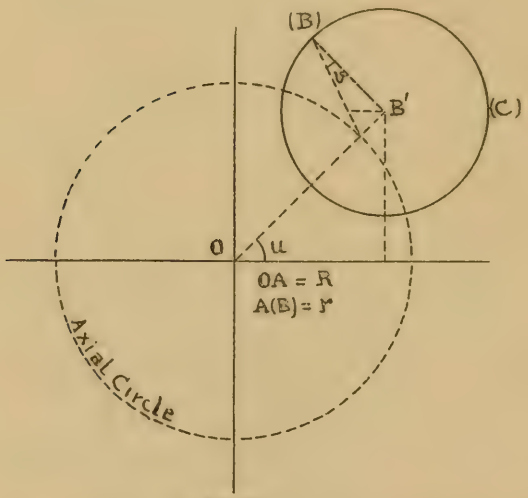


FIG 12.

R and r are respectively the radii of the axial circle and of a meridian of the torus. As it is well known, by means of the expressions (2), the points of the torus are conformally transformed into the points of a rectangle. Considering exclusively the case where $\mathrm{R} > r$, the integral v_1 has the value

(1) A preliminary statement concerning orthographic loxodromics on a torus was made by the author in a paper, "Ueber orthogonale Systeme und einige technische Anwendungen," which appeared in the program of the Polytechnic of Biel, 1898.
An article treating of the same subject appeared in the American Math. Monthly, Vol. VI. p. 135-138.

$$v_1=\frac{r}{\sqrt{R^2-r^2}}\sin^{-1}\left(\frac{1+\frac{R}{r}\sin v}{\frac{R}{r}+\sin v}\right). \tag{3}$$

The values of u and v can each vary from 0 to 2π. In this interval u_1 varies from 0 to 2π and according to the expression (3), v_1 from

$$\frac{r}{\sqrt{R^2-r^2}}\sin^{-1}\frac{r}{R} \text{ to } \frac{r}{\sqrt{R^2-r_2}}\left[\sin^{-1}\frac{R}{r}+2\pi\right].$$

The sides of the primitive rectangle are therefore

$$w_1=2\pi \text{ and } w_2=\frac{r2\pi}{\sqrt{R^2-r^2}}. \tag{4}$$

To the lines $u_1=$const., $v_1=$const. which are respectively parallel to the sides of the rectangle (w_1w_2), correspond the meridians and parallels of the torus. Conversely to the meridians $u=$const., $v=$const., correspond in the rectangle lines parallel to the sides w_1 and w_2 respectively.

Evidently these lines form orthogonal systems. Putting $z=u_1+iv_1$ and designating by α and β two complex quantities $\alpha=p-iq$, $\beta=r(1+i)$, the function

$$\phi+i\psi=\alpha z+\beta,$$

represents also an orthogonal system in the (u_1v_1) – plane. There is

$$\begin{aligned}\phi&=au_1+bv_1+c,\\ \psi&=-bu_1+av_1+c,\end{aligned} \tag{5}$$

Which represent two perpendicular pencils of parallel rays. On the torus they correspond to a system of orthogonal loxodromics.

3. Among the loxodromics of the torus I shall consider those that close after a certain number of revolutions about the axis and the axial circle of the torus. For this purpose consider the elliptic integral

$$z' = \int_0^z \frac{dz}{\sqrt{(1-z^2)\ (1-k^2z^2)}}, \tag{6}$$

by which the positive part of the z-plane is conformally mapped into the rectangle of the z'-plane, whose sides correspond to the periods of the elliptic function

$$z = \mathrm{sn}z'.^{1} \tag{7}$$

In order that the sides of this rectangle be w_1 and w_2, the modulus k of (6) must be chosen

$$k = 4\sqrt{q}\left[\frac{(1+q^2)\ (1+q^4)\ (1+q^6)\ \ldots}{(1+q)\ (1+q^3)\ (1+q^5)\ \ldots}\right]^4,$$

where $q = e^{-\frac{2\pi w_2}{w_1}}$.

By this determination the periods of $z = \mathrm{sn}z'$ become 2π and $\frac{ir2\pi}{\sqrt{R^2-r^2}}$, the general period

$$2\pi\left(m + in\frac{r}{\sqrt{R^2-r^2}}\right). \tag{8}$$

The torus may be considered as a continuous deformation of the doubly-sheeted Riemann surface with the branch points $\pm 1, \pm\frac{1}{k}$, and represents the surface for all doubly periodic functions with the periods (8).

If now in the z' plane, Fig. 13, two lines are drawn from the origin O to two points of the period parallelograms; one to the point

$$\left(2m\pi,\ 2ni\pi\frac{r}{\sqrt{R^2-r^2}}\right),$$

the other to the point

$$\left(2m_1\pi,\ 2n_1i\pi\frac{r}{\sqrt{R^2-r^2}}\right),$$

(1) See F. Klein, "Über Riemann's Theorie der Algebraischen Funktionen und ihrer Integrale," pp. 50-55.

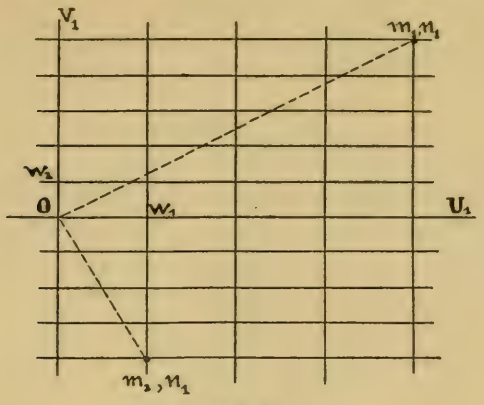


FIG. 13.

then the corresponding lines on the torus are closed loxodromics, and the trigonometric tangent included by them is easily found as

$$\frac{r\left(\frac{n}{m}+\frac{n_1}{m_1}\right)}{1-\frac{n}{m}\cdot\frac{n_1}{m_1}\cdot\frac{r^2}{R^2-r^2}}.$$

The condition for a right angle is

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{R^2-r^2}{r^2}. \tag{9}$$

This must be a rational fraction. Hence, $\frac{r}{R}$ must be rational.

In this case the loxodromics on the torus are orthogonal. Evidently, m gives the number of revolutions of the loxodromic about the axis (perpendicular to the planes of parallel circles) and n the number of revolutions about the axial circle of the torus. Hence the theorem:—

If $\frac{R}{r}$ *is a rational number and if a loxodromic on the corresponding torus closes after m revolutions about the axis and n revolutions about the axial circle, then every orthogonal loxodromic*

closes and the corresponding numbers m_1 *and* n_1 *of revolutions about the same axes are connected to* m *and* n *by the relation*

$$\frac{n}{m} \cdot \frac{n_1}{m_1} = \frac{R^2 - r^2}{r^2}.$$

4. To discuss the nature of the curves on the torus, assume a closed loxodromic, characterized by the ratio $\frac{n}{m}$. In the z'-plane the equation of the corresponding straight line is

$$v_1 = \frac{n}{m} \frac{r}{\sqrt{R^2 - r^2}} u_1,$$

but according to (3),

$$v_1 = \frac{r}{\sqrt{R^2 - r^2}} \sin^{-1} \frac{r + R \sin v}{R + r \sin v},$$

hence

$$\sin \frac{n}{m} u = \frac{r + R \sin v}{R + r \sin v}. \tag{10}$$

In Cartesian coördinates $sin\ u = \frac{y}{\sqrt{x^2+y^2}}$, $sin\ v = \frac{\sqrt{x^2+y^2} - R}{r}$.

Substituting these expressions in (10), the equation of the projection of the loxodromic upon the xy-plane is obtained. Now $sin \frac{n}{m} u$ may be expressed algebraically in terms of $sin\ u$, hence in terms of $\frac{y}{\sqrt{x^2+y^2}}$. The projection of the loxodromic is therefore an algebraic curve. As the torus is an algebraic surface, the theorem follows:—

All closed loxodromics of a torus are algebraic curves.

As an example take $n = 1$, $m = 1$, then the equation of the projection becomes

$$(R^2 - r^2 + ry)^2 = R^2(x^2 + y^2),$$

which represents an ellipse. The corresponding loxodromic itself is a circle as may be easily concluded. Hence the theorem:—

The loxodromics of a torus which turn once about the axis and once about the axial circle are the circles cut out by the double-tangent planes of the torus.

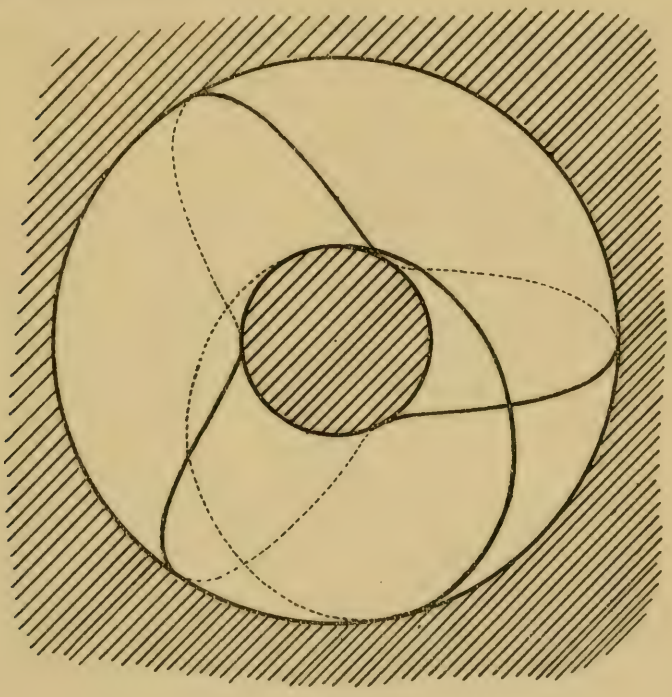


Fig. 14

The numbers of the orthogonal loxodromic are $n=3$, $m=1$ and its projection is a curve of the 6th order, as is shown in Fig. 14. *In general if* $m=1$, *then the projected loxodromic is of the* $2n$*th order, and the loxodromic itself of the* $4n$*th order.*

A proof of this theorem will appear in the December number of the American Mathematical Monthly.

September, 1902.

DESIGN OF FIXED ENDED ARCHES BY THE ELASTIC THEORY.

CHAS. DERLETH, JR.

In the following pages the writer intends to explain the methods for the designing of fixed ended masonry, steel and concrete-steel arches by the elastic theory. The essential parts of the theory involved will be developed and the latest graphical applications given. Designs showing the actual methods used in advanced engineering offices will complete the article.

THEORY.[1]

The ends of an arched rib or girder are considered fixed when the angles formed by their center lines with any fixed line remain unchanged without loading and during the application of varying loads. The piers or abutments are assumed to be immovable under all changes of loading; therefore the total summation of strains at any given distance from the neutral surface of the rib taken throughout the entire length of a fibre must be equal to zero.

Hence:— $$\Sigma_{V}^{V'} nP'z=0. \qquad (1)$$

Here, P' is the strain of extension or compression per unit length of fibre parallel to the neutral surface when that fibre is distant unity from the surface; z is the distance of any fibre from the neutral surface; and n represents the lengths of parts into which that fibre has been divided. The summation sign $\Sigma_{V}^{V'}$ indicates that the sum of all the products like $nP'z$ are to be taken and its limits V and V′ that the sum is to be taken for the entire length of fibre.

(1) See the treatise by Prof. W. H. Burr, "Stresses in Bridges and Roof Trusses," published by John Wiley & Sons.

There are as many terms in the summation as there are n divisions in the fibre. If n becomes a differential the summation becomes an integration.

For a fibre at unit's distance from the neutral surface, z is equal to unity, hence:—

$$\Sigma_{V'}^{V} nP' = 0. \tag{2}$$

From the principles of the common theory of flexure:—

$$P' = \frac{M}{EI}. \tag{3}$$

M is the bending moment at any section in the rib, I the moment of inertia of that section with respect to an axis at right angles to the neutral curve and in the neutral surface of the rib, and E the coefficient of elasticity of the material of the rib.

Therefore:— $$\Sigma_{V'}^{V} nP' = \Sigma_{V'}^{V} \frac{nM}{EI} = 0. \tag{4}$$

If the lengths n, measured on the neutral curve, are selected so that $\frac{n}{I}$ is a constant for all parts into which the rib is divided and if at the same time E is constant, then $\frac{n}{EI}$ is constant, and:—

$$\Sigma_{V'}^{V} nP' = \Sigma_{V'}^{V} \frac{nM}{EI} = \frac{n}{EI} \Sigma_{V'}^{V} M = 0. \tag{5}$$

Therefore:— $$\Sigma_{V'}^{V} M = 0. \tag{6}$$

It follows, therefore, when $\frac{n}{EI}$ is constant that the sum of the bending moments throughout the entire length of the rib must be zero. When the cross section of a rib varies, n must vary. For ribs of constant cross section n becomes constant.

Again since the points V and V′, see plate I, are fixed, while the rib may deflect and distort in any manner between these points, the

total sum of deflections, both horizontal and vertical, for the entire length of any fibre of the rib, must be equal to zero. By the common theory of flexure it can be shown that:—

$$D_v = \Sigma_{V'}^{V} nP'x = \Sigma_{V'}^{V} \frac{nMx}{EI}, \tag{7}$$

$$D_h = \Sigma_{V'}^{V} nP'y = \Sigma_{V'}^{V} \frac{nMy}{EI}. \tag{8}$$

Here D_v and D_h represent the total vertical and horizontal deflections respectively for any given segment or part of a beam or rib depending upon what limits of summation are taken. n, P′, M, E and I have the same significance as above while x and y are the rectangular coördinates of any section. In plate I the origin of coördinates is taken at V′ and the axis of X coincides with VV′, the chord of the neutral curve of the rib.

If the total summation of deflections for the entire length of rib be taken, we have:—

$$D_v = \Sigma_{V'}^{V} nP'x = \frac{n}{EI} \Sigma_{V'}^{V} Mx = 0, \tag{9}$$

$$D_h = \Sigma_{V'}^{V} nP'y = \frac{n}{EI} \Sigma_{V'}^{V} My = 0. \tag{10}$$

Hence:—

$$\Sigma_{V'}^{V} Mx = 0, \tag{11}$$

$$\Sigma_{V'}^{V} My = 0. \tag{12}$$

From equations (11) and (12) it follows that if the total sums be taken of the products obtained by multiplying the bending moments at different sections respectively by the coördinates of such sections that such sums must be equal to zero.

When the ribs are assumed continuous with fixed ends the conditions therefore upon which the design depends are:—

$$\left.\begin{aligned} &\Sigma M=0,\\ &\Sigma Mx=0,\\ &\Sigma My=0,\\ &\frac{n}{EI}=\text{a constant.} \end{aligned}\right\} \qquad (13)$$

Every polygon similar to mBIJKm' or K_1EFGHK_2 is an equilibrium polygon for some loading. Their vertical interscepts in every case are proportional to moments. Of all these polygons, one, namely K_1EFGHK_2 has the same closing line and the same horizontal thrust as the rib. This polygon is known as the true equilibrium frame.

It can be shown from the principles of equilibrium frames (graphic statics) that the bending moments at all points in any arched rib are proportional to the vertical interscepts at such points between the rib neutral curve and the true equilibrium frame. This principle holds both for free and fixed ended ribs. Let M be the bending moment at any point in an arched rib. Let M_1 be the moment intercepts for the true frame K_1EFGHK_2 and M_{11} the interscepts for the rib. Then from what has just been stated:—

$$M=M_1-M_{11}. \qquad (14)$$

Also:—

$$Mx=M_1x-M_{11}x, \qquad (15)$$

$$My=M_1y-M_{11}y. \qquad (16)$$

Consequently:—

$$\Sigma_V^V, Mx=\Sigma_V^V, M_1x-\Sigma_V^V, M_{11}x=0, \qquad (17)$$

$$\Sigma_V^V, My=\Sigma_V^V, M_1y-\Sigma_V^V, M_{11}y=0. \qquad (18)$$

The last relation, equation (18), is used in developing the deflection polygons for the arch rib and the trial frame in order to determine the true pole distance and true interscepts of the true frame.

In designing fixed ended arches by the elastic theory the procedure in applying the above analysis is as follows:—a true equilibrium frame must be found and also a closing line common to it and the

rib which will satisfy the four conditions expressed by equations (13). The pole distance of the true force diagram gives the magnitude of the horizontal arch thrust, the rib closing line gives the line of action of this thrust, and the vertical interscepts between the true frame and the rib are proportional to the bending moments in the rib. These interscepts multiplied by the horizontal thrust give the arch bending moments while the proper components of the rays in the true force diagram give the normal thrusts and the shears at all sections of the rib. The rays of the true force diagram to scale represent the resultant thrusts at the various sections of the rib.

Application.

Concrete-steel structures are becoming all-important to the engineer and before many years of the Twentieth Century have passed, structures of this type will quite generally take the places of those of pure masonry. Monolithic masses of concrete are now used in many ways instead of stone. Retaining walls, bridge piers, culverts, small arch bridges, floors, dams and innumerable other engineering things are now made entirely of concrete and mortar. Even ornaments like balusters and mouldings are made of these materials. In fact highway bridges have lately been built in parks and private estates of this country in which the entire ornamented surfaces are of concrete finished with mortar.

Concrete, like masonry, can withstand great compressive loads but is weak in tension. Where large tensile stresses must be carried concrete, however, can be reinforced and strengthened by embedding in the mass suitable steel work. In such combined structures it is assumed that the concrete masses resist mainly compressive stresses while the steel reduces the tension in the artificial stone to proper low limits or takes the tensile stresses entirely. Where steel work is entirely and securely embedded in concrete it is not acted upon by water and the corrosive agencies of the atmosphere and like wood permanently under water, steel within concrete remains in good condition indefinitely. Concrete-steel structures properly designed may therefore be assumed to be as durable as masonry structures, while

the lighter weight and lesser cost of the former make them much more desirable. Concrete and steel further seem to have very nearly the same coefficient of expansion. Under varying temperatures the combination may be expected to act as one.

The design here considered will for the above reasons and for one other given below be in concrete-steel.

The methods for finding the stresses in fixed ended arches are the same for different materials but in proportioning the rib sections a modification of the formulas of flexure is necessary in cases where concrete-steel is employed.

During the winter of 1899-1900 the writer had the good fortune to be associated with Prof. Wm. H. Burr, of Columbia University, New York, and at that time assisted him in preparing designs for a proposed memorial bridge across the Potomac River at Washington, D. C. Three different designs were prepared and all contained concrete-steel arches. Some of these arches were semi-circular, some circular-segmental, and others semi-elliptical. Their spans ranged from 40 to 283 feet.

In the following design the 192-foot main arch span of Design No. II for the Memorial Bridge will be considered.

This arch has a circular-segmental intrados, radius 173.4 feet, the clear span at the springing line being 192 feet, and the rise of the intrados 29 feet. The extrados is also an arc of a circle but is not concentric with the intrados. Its radius is 213.8 feet. The assumed depth of rib at the crown is 40 inches, that of the springing lines is 100 inches. The center line or neutral curve of the rib is an arc of a circle and is drawn midway between the intrados and extrados. Its radius is 192.0 feet.

The ribs are of concrete in which are embedded masses of steel near both the intrados and the extrados. From sections 1 to 18, throughout the main part of the rib, bars of steel $8''x\frac{7}{8}''$ spaced 2′ 8″ center to center are used in both the top and bottom of the concrete mass. In no case does the steel come nearer to the surface of the concrete than 2 inches. At the springing points and near the ends of the rib heavier steel work is used on account of the higher tensile

and compressive stresses developed there. Circular-segmental arches of low rise generally give considerable tension at the springing joints. In this design the greatest tensions in the rib occur at these places. Latticed girders, therefore, strengthen the concrete in the vicinity of the springing points. The flanges are each composed of 2—6″x3½″ x¾″ angles, 1—13″x¾″ cover plate and 1—13″x⅝″ side plate. The stiffeners and web members are 3½″x3″x⅜″ angles.

The concrete of the ribs is composed of 1—portland cement, 2—sand and 4—broken stone to pass a 2-inch ring and be held on a 1-inch screen.

The original design for this arch contemplated an asphalt roadway supported on concrete floor arches, with the floor arches held by I beam floor beams. The spandrel posts were I beams braced by steel angles. In the revised design for which this article gives the arch analysis the roadway and floor system remained unchanged, but the steel spandrel work of the earlier plans was replaced by concrete posts or pillars connected longitudinally and transveresly by concrete walls and arches. The ribs of the revised design were therefore heavier than in the original drawings.

The calculations herewith are given for one foot width of rib. The dead loads of the spandrel and rib are taken at 150 lbs. per cu. ft., the dead load of the roadway and floor together at 170 lbs. per sq. ft. The live load assumed is 200 lbs. per sq. ft. of roadway plan, comes on the bridge from left to right and covers about five-eighths of the span, or sections V to 6. See Plate I.

In Plate I the linear scale is 1 inch = 60 ft.; the force scale 1 inch = 20,000 lbs.; and the scale for the deflection polygon, 60.

The rib center line or neutral curve VrV′ is divided by trial into eighteen parts whose lengths n are such that $\frac{n}{EI}$ for each part is the same. Here I is the average moment of inertia of any part and is obviously proportional to d^3 where d is the average depth of rib for that part. E is assumed constant and the same for tension and compression. Hence $\frac{n}{d^3}$ is made a constant for each part. The

line VV′ is the chord of the neutral curve; verticals through the centers of gravity of the parts into which the rib has been divided cut the neutral curve at the points 1 to 18 inclusive. Lines like *a b* above the rib indicate the limits of parts. The numbers at the top of Plate I between these lines give the component and total weights on the different parts of the rib. These weights or loads are assumed to act in the vertical lines through the points V′, 1, ... , 18, V. Table I gives the results of the trial division of the rib into parts of lengths n. It will be noticed that $\frac{n}{d^3}$ is essentially constant.

TABLE I.

Point	d In Feet	d^3 In Feet	n In Feet	$\frac{n}{d^3}$
V′—V	----	----	----	----
1—18	6.75	307.5	39.5	0.129
2—17	4.82	112.0	14.0	0.125
3—16	4.25	76.8	9.75	0.127
4—15	3.92	60.0	7.75	0.129
5—14	3.73	51.5	6.6	0.128
6—13	3.60	46.6	5.9	0.127
7—12	3.50	42.9	5.6	0.130
8—11	3.42	40.0	5.5	0.137
9—10	3.38	38.6	5.3	0.137

For convenience a load line AB is laid off upon the vertical through the left springing point. The parts of the load line, A1, 12, etc., represent to scale the total loads dead and live acting at the points 1, 2, etc., of the rib. A pole is selected at any point C whose pole distance = 210,000 lbs., and rays CA, C1, etc., are drawn to the points A, 1, etc., of the load line. The trial polygon BIJKB is

then drawn in the following manner: From B draw BI parallel to the ray CB to terminate in a point I upon the line of action I—18 of the load acting at the centroid of the part 18 of the rib. From I draw the side IJ parallel to the ray C—17 to meet the line J—17 in the point E, etc. Finally connect the points B and K.

Lines like Ic in the polygon BIJKB or $t2$ in the rib are called interscepts while cK and 2V′ are known as the lever arms respectively of the interscepts. Ic and $t2$ are proportional to bending moments. Hence products like $Ic \times c$K represent quantities like Mx. Similarly Ic multiplied by a vertical ordinate would represent a quantity like My.

We must now refer to Table II. Column 1 indicates the point or part of the rib to which the figures to the right refer. Column 2 gives the arch interscepts in feet. They are the vertical interscepts like $t2$ at the respective points of the rib between the chord VV′ and the neutral curve. They are scaled from Plate I.

Since $\Sigma M=0$ for fixed ended arches the closing line K_1K_2 of the rib can now be drawn. $K_1V=K_2V'=$the average of the sum of the arch interscepts$=21.782$ feet.

This closing line K_1K_2 satisfies for the rib the conditions $\frac{n}{EI}=$ constant, $\Sigma M=0$, $\Sigma Mx=0$, and $\Sigma My=0$; because the rib neutral curve is symmetrical both with respect to the vertical and horizontal and because K_1K_2 is horizontal. It is the line of action for the horizontal thrust in the rib. It is also the closing line of the true or equivalent equilibrium frame EFGH.

The trial polygon BIJKB is now operated upon to find the true pole D of the true equilibrium frame K_1K_2HGFE. Columns 3 and 4 of Table II give respectively the interscepts and lever arms of the points of the trial polygon. These distances are scaled from Plate I.

If this trial polygon were the true polygon, a closing line could be found which would make the following relations hold:—

$$\left.\begin{aligned}\Sigma M&=0.\\ \Sigma Mx&=0.\\ \Sigma My&=0.\end{aligned}\right\}$$

A closing line mm' can be found which will make $\Sigma M=0$ and $\Sigma Mx=0$, but unless the assumed pole C happens to be the true pole D (which is not likely) the condition $\Sigma My=0$ cannot be satisfied by the trial polygon.

$Bn=Kn'=$ the average of the trial polygon interscepts $=23.524$ feet. The line nn' might therefore be the true closing line of the trial polygon because since the vertical interscepts between nn' and the broken line BIJK represent bending moments, their total sum by construction is zero. Column 5 of Table II gives quantities corresponding to Mx; they are found by multiplying the interscepts of column 3 by the lever arms of column 4.

We must now find a line mm', the true closing line of the trial polygon, which will make not only $\Sigma M=0$ but also $\Sigma Mx=0$. It sometimes happens that nn' coincides with mm' but not usually.

Draw the diagonal nK in the parallelogram nn'KB dividing it into the triangles nn'K and nKB. Scale the interscepts in these triangles corresponding to the points of the rib. Thus we get column 6 of Table II. These interscepts could better be found by the relations of similar triangles. For example

$$fg=ed=\frac{n\mathrm{B}}{\text{span}}x=\frac{23.524\times 16.29}{192}.$$

Here nB and the span are constants for all interscepts and x is the lever arm for each interscept in turn. One setting of a slide rule can therefore give all the figures in column 6.

Column 7 is found by multiplying the interscepts of column 6 by the lever arms of column 4 and gives quantities similar to moments Mx. Find the sum of column 7.

Divide the sum of column 5 by the sum of column 3. This gives $x'=$ the distance of the line RT from the right springing point V'. In the line RT is found the centroid of the interscepts of the trial polygon BIJKB.

Divide the sum of column 7 by the sum of column 6. The quotient, x'', gives the distance of the centroid of the interscepts of the triangle nKB from the right springing point. The centroid of the

interscepts of the triangle nn'K is an equal distance from the left springing point. The lines UW and XY contain these centroids respectively.

Next find the horizontal distance x''' between the lines UW and XY and also their distances x^{iv} and x^{v} from RT. Thus $x^{iv}=14.957$ feet, $x^{v}=16.063$ feet, and $x'''=31.02$ feet.

Bm is the left end ordinate of the true closing line mm' of the trial polygon mBIJKm', Km' is the right end ordinate. This line mm' makes $\Sigma Mx=0$ in the trial polygon in addition to $\Sigma M=0$ as already accomplished by the line nn'.

$$Bm=\frac{2\times Bn\times x^{v}}{x'''}.$$

$$Km'=\frac{2\times Bn\times x^{iv}}{x'''}.$$

The distances Bm and Km' and therefore the position of the closing line mm' can be found also by graphics. On the line RT lay off RT=Bn=Kn'=23.524 feet. Assume a pole Q and draw rays to the points R and T. QT cuts XY in a point Y. Through T draw TW parallel to QR to cut the line WU in W. Connect W and Y and through the pole Q draw QS parallel to WY to cut the line RT at S. Scale the distances RS and ST. Then locate the closing line mm' by the relations:—

$$\frac{211.71}{18}:RS::Bn:Bm.$$

$$\frac{211.71}{18}:ST::Bn:Km'.$$

The fraction $\frac{211.71}{18}$ is the average of the figures in column 6. The trapezoid mm'KB is now determined. Its interscepts are next desired. Draw the diagonal mK dividing the trapezoid into the triangles mKB and mm'K. Find the interscepts in these triangles by methods similar to those by which the interscepts of the triangles nKB and nn'K were found. The results are given in columns 8

and 9 of Table II. Column 10 gives the interscepts of the trapezoid mm'KB by adding those of columns 8 and 9.[1]

Column 11 gives the interscepts between the closing line mm' and the broken line BIJK found by taking the algebraic sum of columns 10 and 3, considering the moments of column 10 negative and those of column 3 positive.

Column 12 gives the interscepts between the closing line K_1K_2 and the neutral curve of the rib found by taking the algebraic sum of each of the figures in column 2 and the constant interscept $VK_1 = V'K_2$ to the closing line K_1K_2, considering figures in column 2 positive and the common interscept negative.

Columns 11 and 12 give lengths proportional to the true bending moments at the various points of the arched rib and the trial polygonal frame respectively. The algebraic sum of the figures in these columns in each case should be zero since the closing lines K_1K_2 of the rib and mm' of the trial equilibrium frame each satisfy the conditions, $\frac{n}{EI}$ =constant, EM=0 and EMx=0.

The true equivalent equilibrium frame however must also satisfy with the rib the condition $\Sigma My=0$. We must therefore change the position of the assumed pole C to a point D such that the corresponding frame K_1EFGHK_2 may have the closing line K_1K_2 and satisfy the last condition. When this true equilibrium polygon for the rib under the assumed loading has been determined the vertical interscepts between it and the rib's neutral curve will be proportional to the actual bending moments at all points in the rib.

Through C draw CZ parallel to mm' to cut the load line AB in the point Z. At Z erect the perpendicular ZD to the load line. The pole of the true equilibrium frame must lie on this line ZD.

Consider the equation:—

$$\Sigma_V^V, M_I y - \Sigma_V^V, M_{II} y = 0. \qquad (18)$$

$\Sigma_V^V, M_{II} y = 0$ because the rib closing line K_1K_2 has been drawn

(1) The interscepts of the trapezoid can be scaled directly from Plate I; the above method is simple and generally more accurate.

to satisfy that result. Hence the true frame must be so formed as to make $\Sigma \frac{V}{V'} M_1 y = 0$.

Since $\frac{n}{EI}$ is constant $\Sigma \frac{V}{V'} M_1 y$ is also equal to the total horizontal deflection of the true frame between the limits V and V′. The same is true for the rib for the expression $\Sigma \frac{V}{V'} M_{11} y$. See equations (9) and (10). If other limits than V and V′ be chosen, $\Sigma M_1 y$ and $\Sigma M_{11} y$ then represent the total horizontal deflections for the true frame and the rib respectively between those limits. Neither expression need be zero. But they must be equal.

Consider the equations:—

$$M = \Sigma P x, \tag{19}$$

$$D_h = \frac{n}{EI} \Sigma M y. \tag{20}$$

When forces are laid off on a load line and a pole is taken and a funicular polygon is drawn in the space diagram, by graphics, it can be shown that the vertical interscepts in the funicular polygon are proportional to its moments. Hence, by analogy, see equations (19) and (20), if a series of moments be laid off to scale upon a line and be treated like forces, the interscepts in the corresponding frame will be proportional to deflections. Frames so drawn are termed deflection polygons.

Column 13 is found by taking the algebraic sums of the figures in column 12 in the following way: (1+18), (1+18)+(2+17), etc. Column 14 is obtained in like manner from column 11. These are the moment quantities M of equation (20) to be laid off on the so-called load lines of the deflection polygons.

Let *rv* be the load line for the deflection polygon of the trial polygon and *rw* for the rib. Select a common pole O. Lay off the moments of column 14, like forces, upon *rv* and those from column 13 upon *rw*. Draw rays from O. The lines of action of the moments (treated as forces) are respectively along the chords of the

rib's neutral curve joining the points 1—18, 2—17, etc. Proceed then to draw the arch deflection polygon P*ls* by the regular methods of graphics statics and likewise construct P*hk* for the trial frame. The closing lines $rk=p$ and $rs=q$ are proportional to $\Sigma M_1 y$ and $\Sigma M_n y$ respectively. If the trial polygon were the true polygon p would equal q. In order that $\Sigma_V^V M_1 y = \Sigma_V^V M_n y = 0$ they must be equal.

By scale: $p=18.83$ and $q=19.33$. Since p is less than q, the quantities M_1 of the trial polygon are too small and the pole distance is too large. This statement is correct when we remember that a pole distance is always inversely proportional to the vertical interscepts of the funicular polygon corresponding to it. Hence the interscepts, column 15, of the true frame are found by multiplying the figures of column 11 by $\frac{q}{p}$ and the true pole distance ZD is found by multiplying the assumed pole distance by $\frac{p}{q}$.

The true pole D can now be located by scale on the line ZD; ZD=204,800 lbs.

Column 16 gives lines proportional to the true bending moments in the rib, found by subtracting column 12 from column 15.

Column 17 gives these bending moments found by multiplying the lengths in feet in column 16 by the true pole distance in pounds.

The true equilibrium frame can now be located by two different methods:—Its vertices E, F, G, etc., can be located by laying off the interscepts of column 16, observing their signs, from the points on the rib's neutral curve; or the frame can be constructed by drawing its sides in order parallel to the rays of the true force diagram ADB. It is well to locate the frame in both ways as it gives a check upon the preceding computations.

The extreme rays DA and DB of the true force diagram represent the resultant thrusts on the abutments. Resolving these thrusts into components respectively parallel and perpendicular to the tangents to the neutral curve of the rib at the respective springing points V′ and V we get the normal thrusts and the shears on the

arch ring sections V′ and V. Thus in the force diagram, for section V, DL is the normal thrust and LB the shear.

The normal thrust, the shear and the resultant thrust for each point of the arch rib are found respectively in columns 18, 19 and 20.

To find these quantities for any point of the rib other than the springing points a slight modification in the above procedure is necessary. For example, take the point F;—draw the ray DN parallel to the chord EG of the true frame and decompose it into components DM and MN parallel and perpendicular respectively to the tangent to the neutral curve of the rib at the point 18. Then DN is the resultant and DM the normal thrust while MN is the shear at section 18. The same method is used for all other points, 17 to 1 inclusive.

We now have completely obtained the stresses in the rib, columns 17, 18, 19 and 20 giving everything. It is next in order to inquire whether the assumed rib is too strong or too weak.

Column 23 gives the moments of inertia of all rib sections in inch dimensions per inch width of rib. The allowable intensity of compression in concrete will be taken at 450 lbs. per square inch and the allowable tension at 60 lbs. per square inch. 20,000 lbs. per square inch for both tension and compression might be allowed in the steel work. It will be found, however, that the steel from the nature of the combination of steel and concrete cannot be stressed above about one-half this allowable amount.

The stresses in the extreme fibres of the concrete and the steel of concrete-steel ribs of symmetrical section are found by the formulas:—

$$S_1=\frac{T}{A_1+20A_2}\pm\frac{6d_1M}{I_1+20I_2}, \tag{21}$$

$$S_2=\frac{20T}{A_1+20A_2}\pm\frac{120d_2M}{I_1+20I_2}. \tag{22}$$

In these equations:—

S_1=maximum intensities of tension and compression in the concrete of the rib, in pounds per square inch.

S_2=the maximum intensities of tension and compression in the steel of the rib, in pounds per square inch.

T=total normal thrust in pounds per inch width of rib.

A_1=the area of concrete section in square inches per inch width of rib; it is essentially equal numerically to d because the steel area in most cases is comparatively small.

A_2=area in square inches of the average amount of steel flange section per inch width of rib=area in square inches of a steel rib's flange sections divided by the spacing of steel ribs in inches.

d_1=twice the distance in inches from the common centroid of section to the most remote fibre of concrete=for symmetrical sections, d.

d_2=twice the distance in inches from the common centroid to the outermost fibre of steel.

M=the bending moment in foot pounds per inch width of rib.

I_1=moment of inertia of A_1 with respect to the common neutral axis.

I_2=moment of inertia of A_2 about the same axis.

The stresses in the extreme fibres of concrete ribs with no steel are found by making A_2 and I_2 each zero in equation (21), thus:—

$$S_1=\frac{T}{A_1}\pm\frac{6d_1M}{I_1}. \qquad (23)$$

In these equations the coefficient of elasticity for steel is assumed to be twenty times as great as that for concrete, and for each material the coefficient is considered the same in tension and compression. For steel E=30,000,000 pounds per square inch.

For an unsymmetrical section the equations for S_1 and S_2 are identical in form with those given above, only since the common centroid of the materials of the section is no longer at the mid depth, d_1 and d_2 must respectively be replaced by twice the distances to the most remote fibres of steel and concrete on each side of the neutral axis.

By comparing formulas (21) and (22), it is at once seen that the greatest steel stresses are about twenty times the greatest concrete

stresses. Hence the steel of structures like these is never highly stressed since the greatest stresses in the metal cannot exceed about twenty times the greatest allowable concrete stresses; in this case $20 \times 450 = 9{,}000$ lbs.

By substituting in equation (23), the greatest stresses of compression and tension, without steel, are found in the concrete for the different points of the rib. The results are given in pounds per square inch in columns 24 and 25 of Table II. It is seen, therefore, that tension occurs only at and near the springing points with a maximum of 189 pounds per square inch at V and V′. With the exception of points 1 and 2, the maximum compression exceeds the allowable value of 450 pounds per square inch at all points, the greatest intensities being found at the springing points and at the crown. Steel must therefore be introduced to reduce the compression throughout the rib and to relieve the excessive tension at the springing joints.

The arch, 89 feet in total width, is built of ribs 8 feet wide separated by false ribs or coiffures. It is proposed to insert three steel ribs in each 8-foot concrete rib, making the steel rib spacing 32 inches center to center. Across the crown of the arch from point 1 to point 18 the steel flanges will be $8'' \times \frac{7}{8}''$ bars located as shown in figure 1. From points 1 and 18 respectively, to the springing points and continued well into the abutments, the three steel ribs will have flanges like that shown in figure 2. For figure 1, $20A_2 = 8.75$ square inches; for figure 2, $20A_2 = 38.8$ square inches.

By substituting in equations (21) and (22), the greatest stresses of compression and tension, with steel are found, the results being given in columns 26 and 27 of Table II. The greatest compression, therefore, now occurs at the crown, sections 9 and 10, while the greatest tension occurs at the springing joints, V and V′. The greatest compression in the arch concrete is 428 pounds and the greatest tension 55 pounds per square inch. In columns 26 and 27, the positive sign indicates tension, the negative, compression. An examination of column 19 shows that the shears are very small; the greatest shearing intensity being about 19 pounds per square inch (neglecting the steel) and found at the springing joints.

A calculation like that above for determining the necessary steel, might show that the rib was assumed too light, that is, of too small a depth. Such a case would require an unreasonable amount of metal. On the other hand the original assumed depths of rib might have been too large. If the first proportions for the rib are made carefully and by the dictates of experience, later only slight changes will be necessary in the ring proportions. For the most desirable and economic proportions of steel to concrete, the rib may be either slightly increased or decreased in depths by moving the extrados. This should be done in such a manner as still to keep $\frac{n}{EI}$ constant, and unless the movement of the extrados is quite small, especially for large arches, the computations for the new proportions of ring should be repeated. Large changes not only greatly affect the rib depths and the dead weights of the rib, but they also essentially change the position of the rib's neutral curve and closing line, and thus also the values of its interscepts. Apparently no definite, proper and economic ratio of steel to concrete section can be fixed at the beginning of a computation for any given case.

In this design, between sections 1 and 18, the ratio of steel section to concrete varies from 0.0109 at the crown to 0.0054 near sections 1 and 18; while between these sections and the springing points, the ratio varies from 0.24 to 0.194. Throughout the main part of the arch, therefore, the ratio of steel section to concrete is 1% or less, being only 0.5% at the crown.

The high percentages near the springing lines of course are due to the great stresses occurring there. Circular-segmental arches, as has already been mentioned, are always weak at the abutments and high tensions and compressions generally occur there. Other forms of intrados are often more desirable. Elliptic rings and rings of multicentered curves approaching in form the semi-ellipse much more nearly conform in neutral curve to the shape of the true equilibrium frame. Arches of such forms can be readily proportioned which will take no tension at all and in which steel is only necessary

to reduce compressive intensities and to bind the concrete masses together.

In conclusion it may, therefore, be stated that for properly formed ribs, the amount of steel section required need rarely exceed from 1 to 1.5% of that of the concrete.

The various steel ribs and the flanges of each rib should be thoroughly held together by light steel work. Where heavy flanges must be employed it is desirable to hold them together by a stout web system.

In this article it has been assumed that the greatest stresses are produced by a live load covering five-eights of the span. To make the study complete, computations like the above should also be made in turn for a live load covering one-half, three-quarters and the whole span. The greatest abutment pressures occur with the span wholly loaded, but the accompanying bending moments in the rib are small. The greatest interscepts between the arched rib's neutral curve and the true equilibrium frame are found for a live load covering about five-eights of the span. But the smaller the live load the smaller is the pole distance. Hence for the partial live loading the bending moments in the rib are increased by the larger interscepts but decreased by the smaller pole distance. The exact amount of live load to give the greatest bending moments cannot be stated. Fortunately the live loads are relatively small when compared with the dead loads especially of masonry structures so that a small change in the position of the live load does not greatly change the stresses in the ribs. It is, therefore, generally sufficient to make a calculation for one position of the live load.

End abutments should be designed for the arch completely covered with live load. The corresponding arch thrust in the line K_1K_2 equals then very nearly the pole distance, ZD=204,800 pounds, multiplied by the total load, 289,760 pounds on the arch for a full live load, and divided by the total load AB=275,130 pounds for a five-eighths live load. Thus the pole distance for a full live loading

is very nearly: $ZD_1 = 204{,}800 \times \frac{289{,}760}{275{,}130} = 215{,}000$ pounds.

By laying off a new load line $A_1B_1 = 289{,}760$ pounds, and drawing a perpendicular at its middle point, the pole D_1 can be located on the perpendicular. A ray joining the pole D_1 with either A_1 or B_1 gives in amount and direction the total maximum thrust upon the abutment. The point of application of this thrust only is slightly in doubt but with experience it can be located closely enough for all practical purposes.

Piers between two arched ribs should be designated for one span wholly loaded and the other wholly unloaded to give the greatest eccentricity of pressure on the foundation bed.

For either an abutment or an intermediate pier the resultant pressure on the foundation bed is found by graphically combining the proper arch thrust or thrusts with the dead weight of the abutment or pier, the dead weight acting at the centroid of the foundation mass. If the foundations are in water the dead weight of the pier or abutment may have to be reduced by a buoyant pressure.

The computations in this article were made with an ordinary slide rule and will not be found rigorously exact. They are, however, correct far within the necessary limits of practical accuracy.

Thermal Stresses.

Thermal stresses in the past have been generally ignored in concrete-steel arches. And yet, especially in flat arches, even for small variations of temperature these stresses reach very considerable amounts. They should be computed and if necessary allowed for. For arches of metal, thermal stresses must always be computed.

But since this article has already much exceeded the intended limits of the writer, the discussion and computation of thermal stresses will be given in a future number of this publication.

FIG. 1.

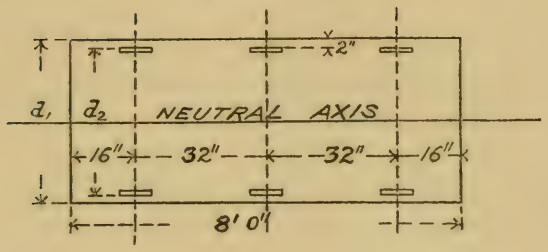

2"
d1
d2
NEUTRAL AXIS
16"
32"
32"
16"
8' 0"

FIG. 2.

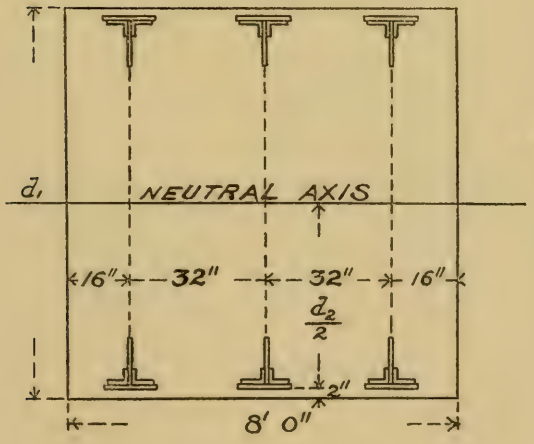

d1
NEUTRAL AXIS
16"
32"
32"
16"
d2/2
2"
8' 0"

TABLE II

1	2	3	4	5	6	7	8	9
Point	Arch Inter-scepts	Trial Polygon			Triangles nn′K and nBK		Interscepts Triangles	
		Inter-scepts	Lever Arms	Moments	Inter-scepts	Moments	mKB′	m′mK
V′	(0)	(0)	0				(0)	(22.69)
1	8.40	10.30	16.29	167.79	2.00	32.58	2.07	20.76
2	18.22	19.22	42.71	820.82	5.23	223.36	5.42	17.64
3	21.18	22.21	54.19	1203.40	6.64	359.80	6.88	16.29
4	22.90	24.06	62.86	1512.30	7.70	484.05	7.98	15.26
5	23.98	25.30	69.85	1767.20	8.56	597.90	8.86	14.43
6	24.72	26.18	76.06	1991.20	9.32	708.82	9.65	13.70
7	25.26	26.83	81.97	2199.40	10.04	822.98	10.40	13.00
8	25.60	27.25	87.66	2388.70	10.74	941.47	11.12	12.33
9	25.77	27.50	93.27	2564.80	11.43	1066.05	11.83	11.66
10	25.77	27.58	98.73	2723.05	12.10	1194.65	12.52	11.02
11	25.60	27.48	104.34	2867.30	12.78	1333.50	13.24	10.36
12	25.26	27.22	110.03	2995.40	13.48	1483.35	13.96	9.69
13	24.72	26.73	115.94	3099.30	14.20	1646.65	14.71	8.99
14	23.98	26.01	122.15	3177.40	14.96	1827.60	15.50	8.25
15	22.90	25.01	129.14	3230.30	15.82	2043.40	16.39	7.43
16	21.18	23.29	137.81	3210.00	16.89	2329.00	17.49	6.40
17	18.22	20.30	149.29	3030.75	18.29	2730.75	18.94	5.05
18	8.40	11.00	175.71	1933.00	21.53	3783.30	22.29	1.92
V	(0)	(0)	192.00		(23.52)		(24.36)	(0)
Sums	392.06	423.47		40,882.11	211.71	23,609.21		

$$K_1V = K_2V' = \frac{392.06}{18} = 21.782 \text{ feet.}$$

$$Bn = n'K = \frac{423.47}{18} = 23.524 \text{ feet; } 2Bn = 2 \times 23.524 = 47.048 \text{ feet.}$$

$$x' = \frac{40882.11}{423.47} = 96.553 \text{ feet; } x'' = \frac{23609.21}{211.71} = 111.51 \text{ feet.}$$

$x^{iv} = 111.51 - 96.553 = 14.957$ feet.

$x^{v} = 111.51 - 95.447 = 16.063$ feet.

$x^{vi} = x^{iv} + x^{v} = 31.02$ feet.

$$Bm = \frac{47.048 \times 16.063}{31.02} = 24.361 \text{ feet.}$$

$$Km' = \frac{47.048 \times 14.597}{31.02} = 22.687 \text{ feet.}$$

$p = 18.83.$

$q = 19.33.$

$$ZD = \frac{210{,}000 \times 18.83}{19.33} = 204{,}600 \text{ lbs.}$$

TABLE II—CONTINUED

10	11	12	13	14	15	16	17	18
Inter-scepts Trape-zoid nn'KB	Trial Polygon True Inter-scepts	Arch True Inter-scepts	Ordinates for Deflection Polygons		Equilib-rium Frame True Inter-scepts	True Bending Moment Inter-scepts	M Ft. Lbs.	Normal Thrust Lbs.
			Arch	Trial Frame				
(22.69)	(−22.69)	(−21.78)			−32.29	− 1.51	−310,000	242,000
22.83	−12.53	−13.38	−26.76	−25.74	−12.86	+ 0.52	+107,000	224,200
23.06	− 3.84	− 3.56	−33.88	−33.27	− 3.94	− 0.38	− 78,000	213,700
23.17	− 0.96	− 0.60	−35.08	−34.83	− 0.99	− 0.39	− 80,000	210,000
23.24	+ 0.82	+ 1.12	−32.84	−32.82	+ 0.84	− 0.28	− 57,500	208,000
23.29	+ 2.01	+ 2.20	−28.44	−28.55	+ 2.05	− 0.15	− 30,800	206,800
23.35	+ 2.83	+ 2.94	−22.56	−22.69	+ 2.91	− 0.03	− 6,150	205,600
23.40	+ 3.43	+ 3.48	−15.60	−15.69	+ 3.52	+ 0.04	+ 8,200	205,000
23.45	+ 3.80	+ 3.82	− 7.96	− 8.01	+ 3.90	+ 0.08	+ 16,400	204,600
23.49	+ 4.01	+ 3.99	+ 0.02	+ 0.04	+ 4.12	+ 0.13	+ 26,700	204,600
23.54	+ 4.04	+ 3.99			+ 4.15	+ 0.16	+ 32,800	204,600
23.60	+ 3.88	+ 3.82			+ 3.98	+ 0.16	+ 32,800	204,600
23.65	+ 3.57	+ 3.48			+ 3.67	+ 0.19	+ 39,000	204,800
23.70	+ 3.03	+ 2.94			+ 3.11	+ 0.17	+ 34,800	205,400
23.75	+ 2.26	+ 2.20			+ 2.32	+ 0.12	+ 24,600	206,200
23.82	+ 1.19	+ 1.12			+ 1.22	+ 0.10	+ 20,460	207,500
23.89	− 0.60	− 0.60			− 0.62	− 0.02	− 4,100	210,000
23.99	− 3.69	− 3.56			− 3.79	− 0.23	− 47,200	215,400
24.21	−13.21	−13.38			−13.56	− 0.18	− 37,000	227,300
(24.36)	(−24.36)	(−21.78)			−25.00	− 3.22	−660,000	249,000

TABLE II—CONCLUDED.

19	20	21		22	23	24	25	26	27
Shear lbs	Resultant Thrust lbs	Assumed Ring Depth		d^3 Inches	I, Inches	Stresses without Steel, lbs per square in.		Stresses with Steel, lbs per square inch	
		Feet	Inches			Compression	Tension	Concrete	Steel
10,500	242,500	8.3	100	1,000,000	83,333	603	198	− 353 + 55	− 6,880 + 920
	224,200	6.7	80.5	521,660	43,300	329		−202	−4,000
4,500	213,800	4.8	57.7	192,100	16,000	450		−370	−7,260
2,500	210,000	4.25	51	132,651	11,080	527		−423	−8,260
4,000	208,000	3.9	46.9	103,162	8,550	526		−419	−8,200
1,700	206,800	3.75	45	91,125	7,570	473		−383	−7,540
3,000	205,600	3.6	43.3	81,183	6,750	504		−402	−7,900
2,700	205,000	3.5	42.2	75,151	6,250	534		−422	−8,280
2,500	204,600	3.4	41	68,921	5,740	532		−420	−8,240
1,800	204.600	3.3	40	64,000	5,330	532		−428	−8,400
900	204,600								
	204,600								
900	204,800								
900	205,400								
	206,200								
1,000	207,500								
4,500	210,000								
9,800	215,600								
7,500	227,800								
23,000	250,000								

Parentheses ( ) in lines V and V′ indicate that the figures which they contain are not included in the sums.

Columns 2 to 20 inclusive, are computed for one foot width of rib.

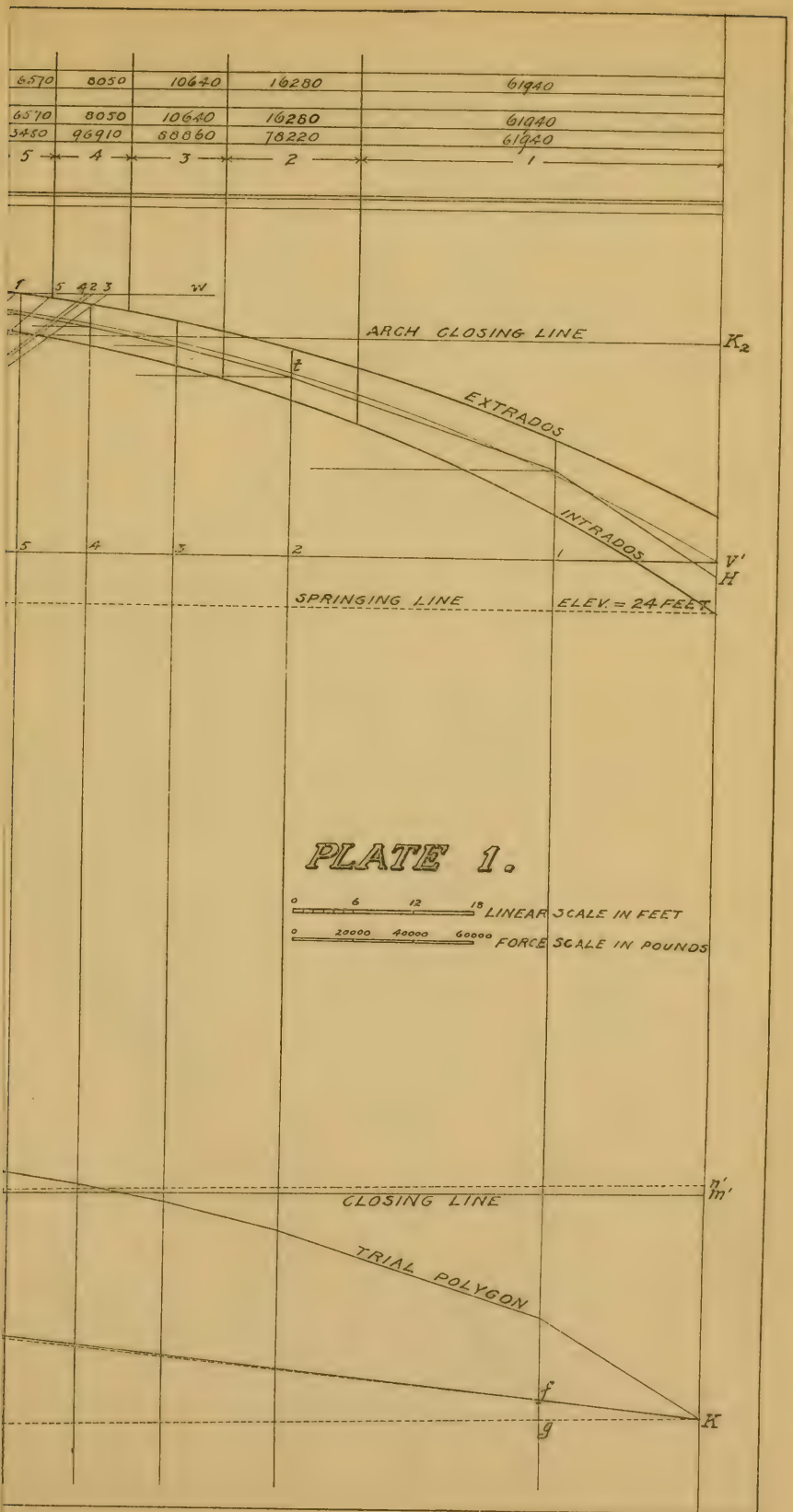
Columns 2 to 16 inclusive, are in feet dimensions.

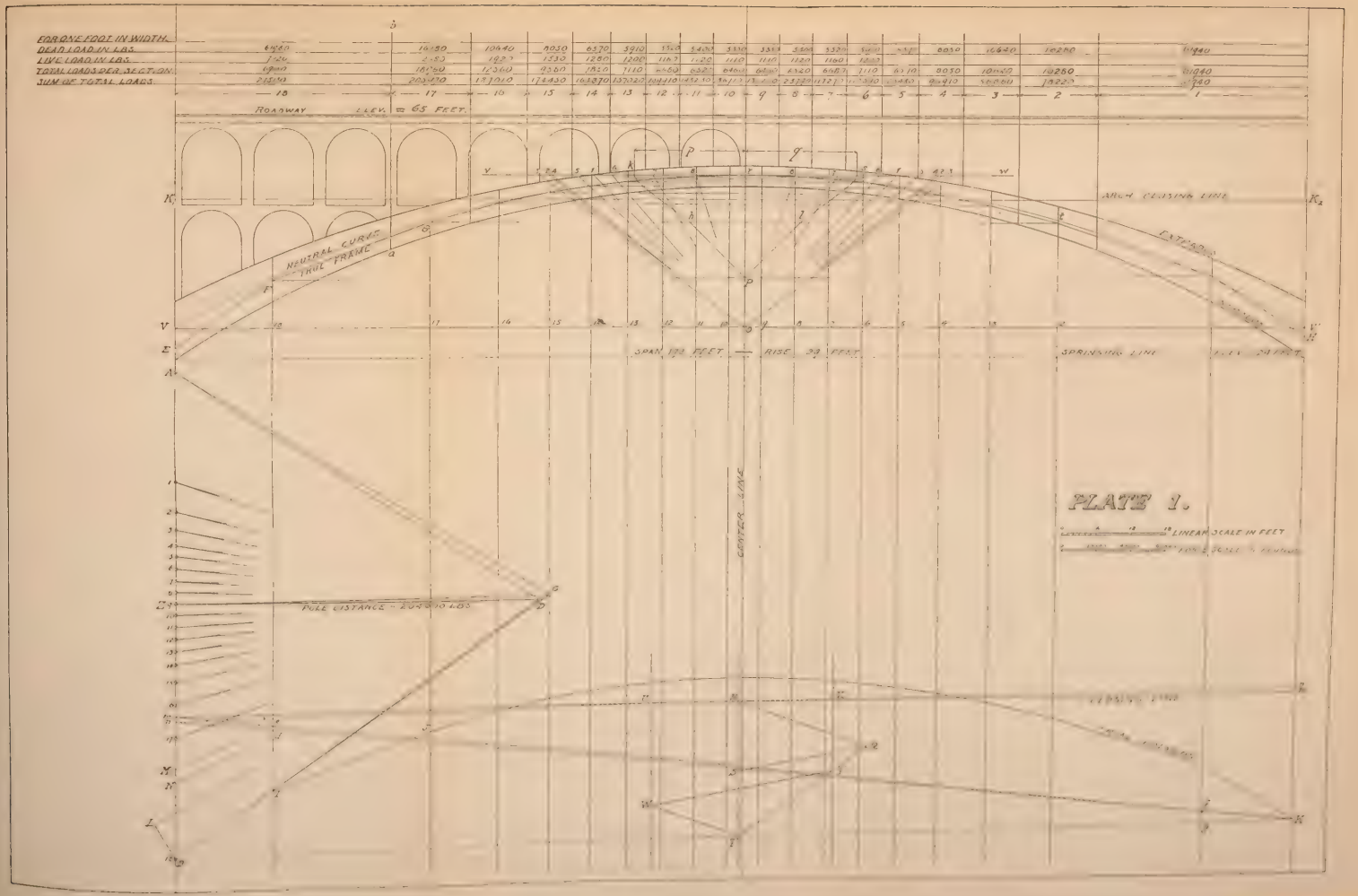
Column 23 gives the moment of inertia for one inch width of rib.

Total load AB on arch one foot wide for five-eights live load = 275,130 lbs.

Total load A_1B_1 on arch one foot wide for whole span loaded = 289,760 lbs.

Pole distance, whole span loaded = 215,000 lbs.


6570 8050 10640 16280 61940
6570 8050 10640 16250 61940
3450 96910 88860 78220 61940
5 4 3 2 1
f 5 4 2 3 w
ARCH CLOSING LINE
K_2
t
EXTRADOS
INTRADOS
5 4 3 2 1
V'
H
SPRINGING LINE
ELEV. = 24 FEET
PLATE 1.
0 6 12 18 LINEAR SCALE IN FEET
0 20000 40000 60000 FORCE SCALE IN POUNDS
n' m'
CLOSING LINE
TRIAL POLYGON
f
K
g


FOR ONE FOOT IN WIDTH.
DEAD LOAD IN LBS.
LIVE LOAD IN LBS.
TOTAL LOADS PER SECTION.
SUM OF TOTAL LOADS.
ROADWAY
ELEV. = 65 FEET.
NEUTRAL CURVE TRIAL FRAME
ARCH CLOSING LINE
SPRINGING LINE
SPAN 120 FEET — RISE 23 FEET
CENTER LINE
POLE DISTANCE = 2,000 LBS.
PLATE I.
LINEAR SCALE IN FEET

ON THE ACTION OF THE HALOGENS AND THE SULPHUR HALIDES UPON PARATOLUQUINOLINE.

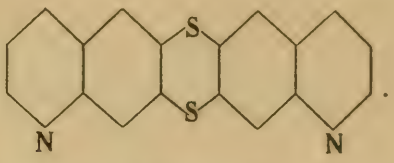
JOHN B. EKELEY*

The action of sulphur and the chlorides of sulphur upon the organic bases, quinoline, orthotoluquinoline, oxyquinoline, isoquinoline, and acridine has been studied by Edinger and his pupils. It has long been known that the introduction of sulphur into cyanogen compounds robs these of their poisonous qualities by the formation of the harmless sulphocyanides. In like manner the investigations of Edinger[1], Edinger and Treupel[2], and Edinger and Arnold[3] show that sulphur has a similar effect upon the organic bases above mentioned. It became of interest to study the action of the chlorides of sulphur upon paratoluquinoline, more especially because, in the investigations of Edinger and his students, in the cases of quinoline and orthotoluquinoline, difficulties were encountered in that, by the elementary analysis, an amount of hydrogen was always found which was too high. This could not be referred back to any impurities which might have been in the compounds analyzed. By the action of S_2Cl_2 upon quinoline a compound containing sulphur was obtained, the sulphur of which was held in combination with unusual tenacity, so much so, in fact, that the compound could be distilled over glowing copper under diminished pressure without decomposition taking place. For this reason and also from a determination of the molecular weight by the lowering of the freezing point method, it was assumed

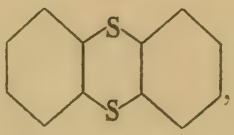
(1) Journal für Prakt. Chemie, Band 54, 340; Band 56, 273.
(2) Münchener Med. Wochenschrift, No. 21, 22, 39, 1900.
(3) Journal für Prakt. Chemie, Band 64, 182, 471.

*The author wishes to express his appreciation to Prof. Dr. Albert Edinger of the University of Freiburg in Baden, under whose direction this investigation was carried out.

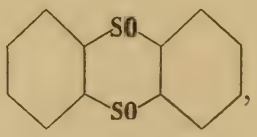
that in the reaction two quinoline molecules had been combined by means of two sulphur atoms in the manner represented by the formula



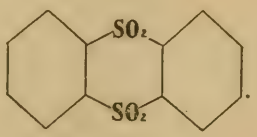
Now Schmidt[1] obtained from S_2Cl_2 and benzene a compound



which Krafft[2] obtained through a different method and named Thianthrene. This body however shows the characteristic properties of a disulphide. On oxidation with nitric acid Krafft obtained a disulphoxide

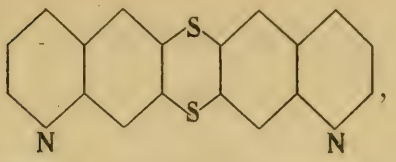


which could be reduced easily to thianthrene by means of zinc and acetic acid. Chromic acid oxidizes thianthrene to a disulphone,

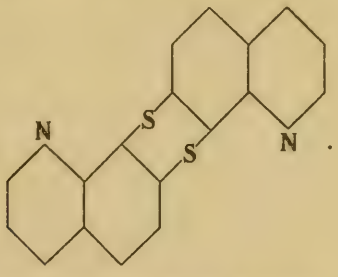


On the other hand the compound obtained by Edinger from quinoline and S_2Cl_2, and to which he gave the name Thiochinanthrene and the constitution

(1) Berliner Berichte, 11. 1168.
(2) Berliner Berichte, 29. 435.



shows entirely different properties. Orthotoluquinoline gave a similar compound, an isomer of thiochinanthrene, since the methyl group is eliminated in the course of the reaction. The constitution according to Edinger would be



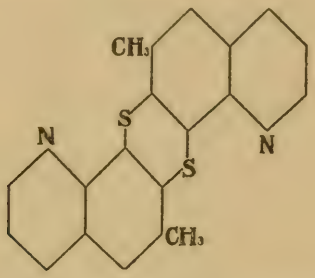
These two thiochinanthrenes show entirely different properties from thianthren and its homologues. By their oxidation no sulphoxides or sulphones can be obtained. The compounds remain either unchanged or are completely oxidized. In one case Edinger obtained nicotinic acid by the oxidation of the isomeric thiochinanthrene.

In the light of the above mentioned researches, it was of interest to determine, first, whether paratoluquinoline would give an analogous compound; second, whether in this case also the methyl group would be eliminated; and, third, if here also a too high percentage of hydrogen would be shown, and, in that case, how the theory disturbing fact could be explained. The following research has, it is believed, given answers to these questions:—

THEORETICAL PART.

The action of the chlorides of sulphur upon aromatic tertiary amines leads to the result, according to Edinger, that the above mentioned sulphur bodies are only then formed when the base acted upon

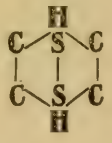
contains a benzene rest having all its central bonds intact. Thus, for example, it was not possible to obtain the compounds in the cases of pyridine, isoquinoline, and oxyquinoline. In the case of paratoluquinoline, one would expect, however, that the reaction would take place, and this was indeed found to be true, since, as will be shown in the experimental part, a sulphur body was obtained which was an exact analogue to those obtained by Edinger, and in which the methyl group was retained. In the light of the previous researches, the following formula would be given to this body:—



The strange fact here also developed that, by the analysis of the compound, in spite of its careful purification by several recrystallizations, a too high percentage of hydrogen to conform to the above formula was found. The possibility had then to be taken into account that an addition of hydrogen in the course of the reaction might have taken place. All experiments calculated to show such an addition of hydrogen in the quinoline molecule were fruitless. It finally became necessary therefore to consider the assumption that an addition of hydrogen had taken place on the sulphur atoms and that consequently, in these compounds, sulphur is tetravalent and not divalent, as heretofore assumed. The most remarkable examples of tetravalent sulphur in organic compounds are the sulphur bases R_3SOH, all of which show an alkaline reaction. Kehrman[1] and Werner[2] have also lately noticed organic compounds in which sulphur shows basic properties. The question therefore came into the

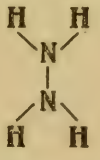
(1) Berliner Berichte, 34. 4172.
(2) Berliner Berichte, 34. 3311.

foreground whether, in these cases also, it could be shown that acids could be added to the sulphur in the compounds. This was, as will be shown, the case; and it seems that we have obtained here for the first time compounds which contain in a ring two sulphur atoms which show basic properties. The name "dithionium bases" might well be given to this series of substances. In the light of what has been said, the following ring—

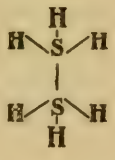


may be considered peculiar to them. They are therefore to be looked upon as substitution products of an at present unknown sulphur analogue of hydrazine whose formula would be S_2H_6. In this four valences are satisfied by valences from four ring-bound carbon atoms, while two hydrogen atoms remain in it intact.

I. Hydrazine

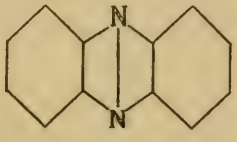


II. Unknown Sulphur Analogue

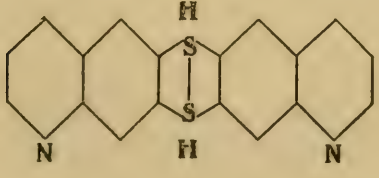


Phenazine can be regarded as a derivative of I, and thiochinanthrene as a derivative of II.

Phenazine



Thiochinanthrene



For these reasons it becomes necessary to modify the former formula for thiochinanthrene by the addition of two hydrogen atoms. As will be shown in the experimental part, this explains the fact that in every case a too large amount of hydrogen was found to agree with the old formulas.

All compounds of this class add four molecules of nitric acid, namely two on the nitrogen of the quinoline and two on the sulphur atoms. Sulphuric acid salts are also formed in that the bases unite with two molecules of the acid. In the cases of hydrochloric and hydrobromic acids, it was not possible to obtain tetrachlorides and tetrabromides. The bases enter into a peculiar reaction with bromine. They all add bromine in acetic acid solution. The bromine compounds thus formed are exceedingly unstable and give up bromine rapidly as soon as the crystals are removed from the mother liquor. Since it is not possible to dry the crystals quickly enough, it is possible only to obtain approximate analyses of them. By heating the bromine products one does not obtain, as might perhaps be expected from the Claus-Collischonn reaction, bromine substitution products; on the other hand the compounds pass gradually, through loss of bromine, over into the hydrobromic acid salts, and finally, through loss of hydrobromic acid, into the original bases. These facts would indicate that the bromine was added to the sulphur atoms and the hydrobromic acid to the nitrogen of the quinoline rest, an assumption which would fully agree with the conclusion arrived at in regard to the constitution of the compounds. In the case of thiochinanthrene, it was possible, by the driving off of bromine, to obtain the hydrobromic acid salt chemically pure, as shown by the analysis.

Experiments were also tried with the purpose of replacing the sulphur hydrogen by means of acetyl. The attempts were all unsuccessful, although at first it was supposed that bodies of this kind were obtained.

In the reaction of SCl_2 upon paratoluquinoline, there are obtained as by-products to the main reaction (which is the formation of the sulphur body) chlorine substitution products. In these products the

methyl group remains unchanged in contrast to what occurs in the reaction with orthotoluquinoline where the methyl group is split off. The following chlorine bodies were obtained: a dichlorparatoluquinoline, melting at 80°—81°, and a trichlorparatoluquinoline, melting at 159°. These could easily be separated by means of hydrochloric acid on account of their difference in basicity.

In a former investigation Edinger[1] has shown that bromine can easily be substituted into the quinoline molecule by means of S_2Br_2. Claus and Müller[2], and Claus and Lang[3] obtained by means of bromine alone, in the case of orthotoluquinoline, products in which the bromine was substituted in the side-chain. By the action of S_2Br_2 upon paratoluquinoline a substitution product was here obtained, in which the bromine entered the quinoline molecule itself. This monobromparatoluquinoline melts at 84°—85°.

Juvalta[4] and Rupp[5] have shown that iodine can easily be substituted into aromatic hydrocarbons by means of fuming sulphuric acid. This method was used to obtain iodine substitution products, inasmuch as frequent attempts to introduce iodine into quinoline by means of the iodide of sulphur have proven useless. By the use of acid containing 50 per cent. SO_3 and the calculated amount of iodine, splendid yields of a di-iodoparatoluquinoline melting at 135° were obtained. In like manner a dibromparatoluquinoline melting at 135° can be made.

EXPERIMENTAL PART.

Action of the chlorides of sulphur upon paratoluquinoline.

A.

Action of S_2Cl_2 on the oil bath at 150°.

Fifty grms. paratoluquinoline were mixed with one hundred grms. S_2Cl_2 in a flask immersed in ice. The mixture was then

(1) Journal für Prakt. Chemie, Band 54, 355.
(2) Freiburger Dissertationen, 1897.
(3) Freiburger Dissertationen, 1898.
(4) Friedlander, Fortschritte der Teerfarbenfabrikation, 2, 93.
(5) Berliner Berichte, 29, 1625.

heated on the oil bath at 150° for four to five hours, a reflex condenser being used. After cooling, the flask was broken, the dark brown solid product was powdered in a mortar and treated with hydrochloric acid on the water bath until the resulting liquid, after repeated treatment, showed only a slight brown color. It is best to use a large flask for this, otherwise a large portion of the solution may be lost by the foaming up of the acid, due to the presence of an excess of S_2Cl_2. After the solution thus obtained had been boiled some minutes with animal charcoal, it was filtered and then treated with a current of steam. The hydrochloric acid salt of the new base crystallizes out on cooling. It is necessary to treat the solution with steam several times in order to obtain the greatest possible amount of the salt. The salt is then filtered, using the suction pump, and afterwards boiled a few minutes with water. It is thus decomposed into acid and base. The base can be obtained pure by dissolving it in acetic acid, heating with animal charcoal, and twice recrystallizing. It crystallizes in small white needles having a melting point of 316°. They dissolve only in mineral acids, in boiling acetic acid, and in boiling xylene.

Out of the original filtrate, it is possible to obtain only slight traces of chlorine substitution products by means of distillation with steam.

Carbon and Hydrogen Determination.

.1717 grms. substance gave .4359 grms. CO_2 and .0753 grms. H_2O.

Computed:	Found:
$C_{20}H_{16}N_2S_2$.	
C=68.98%.	69.25%.
H=4.60%.	4.91%.

Sulphur Determination.

.1803 grms. substance gave .2469 grms. $BaSO_4$.

Computed:	Found:
S=18.40%.	18.80%.

Nitrogen Determination.

.1685 grms. substance gave at 746.7 m. m. and 29° 13.1 c. c. of N.

Computed:	Found:
N=8.04%.	8.27%.

Paratoluthiochinanthrene gives beautifully crystallizing salts with the mineral acids and with picric acid. The sulphuric acid salt crystallizes in two modifications, one anhydrous, gleaming yellow needles, the other containing water of crystallization, scarlet needles. The yellow variety is the unstable one. It separates out first from the solution and changes slowly on standing, or suddenly on filtering, into the scarlet variety. The scarlet variety loses two molecules of water on standing over sulphuric acid in a desiccator, and passes over into the yellow form. This yellow variety changes back to the scarlet by standing in the air or by being pressed by any hard object. This last phenomenon is explained by the fact that it is impossible to remove the last traces of sulphuric acid from the crystals (one can not wash them with water, since they then separate into base and acid); therefore there remains a little somewhat diluted sulphuric acid on the crystals after they are dried in the desiccator. By suddenly pressing the crystals they take this water on from the acid in the form of water of crystallization and thus pass over into the scarlet modification. The hydrochloric acid salt of the corresponding base from orthotoluquinoline shows the same phenomenon, except that in this case the colors are reversed, in that the yellow salt contains water and the scarlet one is anhydrous.

All of these salts are decomposed by heating with water.

Nitric Acid Salt.

By dissolving the base in concentrated nitric acid, one obtains, when the solution cools, long, golden needles, which contain four molecules of nitric acid. The nitric acid was determined not only by titration with standard $Ba(OH)_2$ solution but also by means of a nitrogen determination (Dumas).

Acid Determination.

.1784 grms. substance after boiling with water required 10.4 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, $4HNO_3$=42.00%.	41.72%.

Nitrogen Determination.

.1592 grms. substance gave 20.1 c. c. N. at 736 m. m. and 18°.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, $4HNO_3$, N=14.00%.	N=14.11%.

Sulphuric Acid Salt.

The base was dissolved in hot sulphuric acid, (2 parts water to 1 part acid). From this solution on cooling the salt crystallized out.

Acid Determination.

a. Red Modification.

.1634 grms. substance after boiling with water required for neutralization 10 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, $2H_2SO_4$, $2H_2O$, H_2SO_4=33.79%.	34.05%.

b. Yellow Modification.

.2645 grms. substance after boiling with water required for neutralization 17.4 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, $2H_2SO_4$, H_2SO_4=36.03%.	36 61%.

In spite of the fact that the yellow salt was most carefully analyzed, it was not possible to obtain results which agreed closer than about .6% with what theory required. This has the following cause: The red variety can not be entirely freed from a small amount of

somewhat diluted sulphuric acid which remains on the crystals. The analysis of this variety gives however good results because the extra water which is present compensates the extra amount of sulphuric acid. When the red variety is dried in the desiccator, the water of crystallization is removed, not however that which is present on account of the mechanically adhering acid. A too high amount of sulphuric acid is therefore found, always about .6%. This assumption, that mechanically adhering diluted sulphuric acid influences the analysis, was confirmed by the following water determinations, in which it was found that the yellow variety really contained from .1% to .2% of water.

The determinations were made from samples from three different crystallizations. The following amounts of substance were heated at 110° to constant weight:

Red.	a.	.1491 grms.	lost	.0106 grms.	=	7.11%.
	b.	.2417 "	"	.0201 "	=	8.31%.
	c.	.1841 "	"	.0139 "	=	7.55%.
Yellow.	a′.	.2108 grms.	lost	.0028 grms.	=	1.18%.
	b′.	.1717 "	"	.0034 "	=	1.98%.
	c′.	.2369 "	"	.0030 "	=	1.27%.

Difference between the corresponding red and yellow modification = amount of water found.

$a—a' = 5.93\%$.
$b—b' = 6.33\%$.
$c—c' = 6.27\%$.

Computed:
$2H_2O = 6.20\%$.

From this it seems most probable that the red modification contains two molecules of water of crystallization, while the yellow in the pure state would be anhydrous.

Hydrochloric Acid Salt.

The base dissolves with difficulty in concentrated, easily in dilute, hydrochloric acid. From this solution yellow crystals of the HCl salt separate out. It is better however to dissolve the base in

boiling xylene and then to lead into this solution a stream of dry HCl gas, whereupon the salt is precipitated in the form of microscopic yellow needles. The salt contains two molecules of acid.

Acid Determination.

.1547 grms. substance after boiling with water required 6.5 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
2HCl=17.34%.	17.42%.

Hydrobromic Acid Salt.

The hydrobromic acid salt was prepared in a manner analogous to that of the hydrochloric acid salt. It is also yellow and contains two molecules of acid.

Acid Determination.

.2506 grms. substance after boiling with water required 8.6 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
2HBr=31.61%.	31.58%.

Picric Acid Salt.

When an acetic acid solution of picric acid is added to a hot acetic acid solution of the base, the picric acid salt separates out in the form of yellow needles.

Nitrogen Determination.

.2303 grms. substance gave 28.3 c. c. N. at 737 m. m. and 12.5°.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, $2C_6H_2(NO_2)_3OH$, N=13.89%.	14.11%.

Methiodide.

The base was heated with an excess of methyliodide in a sealed tube at 100°. The resulting methiodide was recrystallized from water and obtained in the form of red needles.

Iodine Determination.

.2097 grms. substance gave by the method of Carius .1570 grams Ag I.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, $2CH_3I$,	
I=40.19%.	40.46%.

Salts of Thiochinanthrene.

Nitric Acid Salt.

The nitric acid salt crystallizes out from a solution of the base in hot concentrated nitric acid. It contains two molecules of water of crystallization. This could not, however, be directly determined, since the salt decomposes between 80° and 100°. By standing in a desiccator the water is not given off.

Nitrogen Determination.

.2443 grms. substance gave 29.1 c. c. N. at 734 m. m. and 17°.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, $4HNO_3$, $2H_2O$,	
N=13.05%.	13.33%.

Acid Determination.

.2510 grms. substance after boiling with water required 14.5 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$4HNO_3$=41.44%.	41.35%.

Sulphuric Acid Salt.

The salt was obtained in a manner similar to that of paratolu-thiochinanthrene. In this case the salt crystallizes without water in the form of red brown needles.

Acid Determination.

.3342 grms. substance after boiling with water required 23 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, $2H_2SO_4$, $2H_2SO_4$=37.98%.	38.31%.

Hydrochloric Acid Salt.

From a hot xylene solution of the base, a stream of dry HCl gas precipitates the HCl salt.

Acid Determination.

.1887 grms. substance after boiling with water required 8.55 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HCl, 2HCl=18.58%.	18.78%.

Hydrobromic Acid Salt.

The hydrobromic acid salt is prepared similarly to that of paratoluthiochinanthrene.

Acid Determination.

.1953 grms. substance after boiling with water required 7.09 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HBr, 2HBr=33.61%.	33.40%.

Salts of the Base from Orthotoluquinoline.

Nitric Acid Salt.

Yellow crystals of the nitric acid salt separate out from a solution of the base in hot concentrated nitric acid. They contain two

molecules of water of crystallization, which, as in the case of the nitric acid salt of thiochinanthrene, can not be determined directly, since the salt decomposes between 80° and 100°.

Nitrogen Determination.

.1542 grms. substance gave 18.2 c. c. N. at 734.5 m. m. and 20.5°.

Computed:	Found:
$C_{18}H_{12}N_2S_2, 4HNO_3, 2H_2O,$	
N=13.05%.	13.04%.

Acid Determination.

.2004 grms. substance after boiling with water required 11.67 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$4HNO_3$=41.44%.	41.67%.

Sulphuric Acid Salt.

From a solution of the base in strong sulphuric acid, the salt crystallizes in the form of red-brown needles.

Acid Determination.

.1674 grms. substance after boiling with water required 11.4 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2, 2H_2SO_4,$	
$2H_2SO_4$=37.98%.	37.91%.

Hydrochloric Acid Salt.

A stream of dry HCl precipitates microscopic needles of the scarlet HCl salt from a hot xylene solution. These change to a yellow modification by standing in the air, water of crystallization being taken on. By slightly heating them they may be changed back to the scarlet form. The yellow form crystallizes out from a solution of the base in aqueous hydrochloric acid.

Acid Determinations.

(a). Scarlet Salt.

.1426 grms. substance after boiling with water required 6.3 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HCl,	
2HCl=18.58%.	18.32%.

(b). Yellow Salt.

.1449 grms. substance after boiling with water required 5.4 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HCl, $2H_2O$,	
2HCl=15.69%.	15.45%.

Water Determination.

.1723 grms. substance heated to constant weight at 100° lost .0138 grms.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HCl, $2H_2O$,	
$2H_2O$=8.39%.	8.01%.

Hydrobromic Acid Salt.

Dry HBr gas precipitates the dark red HBr salt from a hot xylene solution of the base.

Acid Determination.

.1644 grms. substance after boiling with water required 6.05 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HBr,	
2HBr=33.61%.	33.86%.

Bromine Addition Products of the Bases.

(a). From Paratoluthiochinanthrene.

When bromine is allowed to drop into a hot acetic acid solution of the base, a point is soon reached at which suddenly bright red crystals of a bromine addition product separate out. As has before been stated, it is impossible to obtain accurate analyses of these, because, as soon as they are removed from the mother liquor, they begin to give off bromine. A bromine determination showed however that they contain more than four bromine atoms to the molecule. By heating them at 100°, they lose bromine and hydrobromic acid, and there is formed a mixture of the free base and the hydrobromic acid salt. By further heating all bromine is driven off.

Bromine Determination.

.1474 grms. substance gave by the method of Carius .1807 grms. Ag Br.

Computed:	Found:
$C_{20}H_{16}N_2S_2$, 2HBr, Br_4, 6Br=58%.	52.09%.
$C_{20}H_{16}N_2S_2$, 2HBr, Br_2, 4Br=48%.	

(b). From Thiochinanthrene.

A similar experiment with thiochinanthrene gave a beautifully crystallized red bromine body, which also gives off bromine as soon as it is removed from the solution. The compound showed by analysis more than five atoms of bromine. In this case however, the pure HBr salt was obtained by heating the product to constant weight at 115°.

Bromine Determination.

.1512 grms. substance gave .1992 grms. Ag Br.

Computed:	Found:
$C_{18}H_{12}N_2S_2$, 2HBr, Br_4, 6Br=67.6%.	56.07%.
5Br=55.7%.	

Acid Determination in the HBr Salt obtained by heating the Bromine Product to Constant Weight.

.1726 grms. substance after boiling with water required 6.75 c. c. of a .1136 N. $Ba(OH)_2$ solution.

Computed:	Found:
$C_{15}H_{12}N_2S_2$, 2HBr, 2HBr=33.61%.	33.32%.

(c). From Base from Orthotoluquinoline.

As in the two other cases, a red crystalline unstable bromine product was obtained. By heating it at 100° the original red changes gradually to quite another red color, which exactly corresponds to the color of the HBr salt. The decomposition proceeds until the free base is obtained. A bromine determination showed more than five bromine atoms to the molecule.

Bromine Determination.

.1593 grms. substance gave .2207 grms. Ag Br.

Computed:	Found:
$C_{15}H_{12}N_2S_2$, 2HBr, Br_4, 6HBr=67.6%.	58.96%.
5HBr=55.7%.	

B.

Action of SCl_2 on Paratoluquinoline.

Twenty grms. of paratoluquinoline were mixed with thirty grms. of SCl_2, the flask being at the same time cooled with ice. After heating the mixture five hours at 120° upon the oil bath, the semi-liquid product was successively treated with concentrated HCl as long as a brown coloration showed in the solutions. The solution was then heated with animal charcoal and filtered through cotton, whereupon the HCl salt of the sulphur base crystallized out. This was purified as in A.

The filtrate from the HCl salt was then distilled with superheated steam. A mixture of two chlorine substituted paratoluquino-

lines passed over. These were filtered off, and heated with dilute HCl. A dichlor product dissolves leaving a trichlor product unchanged. The filtrate containing the dichlorquinoline was made alkaline with ammonia, whereupon the base separated out. On recrystallizing from alcohol, snow white needles melting at 80° —81° are obtained.

Carbon and Hydrogen Determination.

.1626 grms. substance gave .3383 grms. CO_2 and .0520 grms. H_2O.

Computed:	Found:
$C_{10}H_7NCl_2$,	
C=56.65%.	56.63%.
H=3.30%.	3.58%.

Nitrogen Determination.

.1522 grms. substance gave 8.8 c. c. N. at 751 m. m. and 13°.

Computed:	Found:
N=6.60%.	6.73%.

Chlorine Determination.

.1537 grms. substance gave .2093 grms. Ag Cl.

Computed:	Found:
Cl=33.53%.	33.68%.

The trichlorparatoluquinoline, which was insoluble in dilute HCl, was recrystallized from alcohol. It is white and melts at 159°.

Carbon and Hydrogen Determination.

.1733 grms. substance gave .3114 grms. CO_2 and .0439 grms. H_2O.

Computed:	Found:
$C_{10}H_6NCl_3$,	
C=49.08%.	49.01%.
H=2.44%.	2.83%.

Nitrogen Determination.

.1600 grms. substance gave 8.2 c. c. N. at 738 m. m. and 27.5°.

Computed:	Found:
N=5.68%.	5.47%.

Chlorine Determination.

.1746 grms. substance gave .3068 grms. Ag Cl.

Computed:	Found:
Cl=43.20%.	43.45%.

METHIODIDE.

The base was heated with excess CH_3I in a sealed tube at 100°. The tube was broken and the excess CH_3I evaporated. A recrystallization from alcohol gave red needles of the product.

Nitrogen Determination.

.1914 grms. substance gave 7.4 c. c. N. at 736 m. m. and 19°.

Computed:	Found:
$C_{10}H_7NCl_2$, CH_3I,	
N=3.95%.	4.30%.

PLATINUM DOUBLE SALT OF DICHLORPARATOLUQUINOLINE.

The platinum double salt separates out when platinum chloride is added to the HCl solution of the base.

Platinum Determination.

.1543 grms. of substance dried at 100° and ignited gave .0359 grms. platinum.

Computed:	Found:
$(C_{10}H_7NCl_2)_2H_2PtCl_6$.	
Pt=23.38%.	23.27%.

C.

ACTION OF S_2Br_2 ON PARATOLUQUINOLINE.

Ten grams of paratoluquinoline were mixed with thirty grams S_2Br_2 and heated several hours in the oil bath at 120°. No sulphur base is here formed. The product of the reaction was treated with concentrated HCl, and the solution thus obtained was distilled with superheated steam. A poor yield of a monobromparatoluquinoline

distilled over, which after recrystallization from alcohol melted at 84°—85°. White needles.

Nitrogen Determination.

.1059 grms. substance gave 5.8 c. c. N. at 735 m. m. and 20°.

Computed:	Found:
$C_{10}H_8NBr$,	
N=6.30%.	6.05%.

Bromine Determination.

.1322 grms. gave .1131 grms. Ag Br.

Computed:	Found:
Br=36.03%.	36.42%.

Platinum Double Salt.

Platinum chloride precipitates the double salt from the HCl solution of the base. Yellow needles.

Platinum Determination.

.1202 grms. substance gave .0271 grms. platinum on ignition.

Computed:	Found:
$(C_{10}H_8NBr)_2H_2PtCl_6$,	
Pt=22.83%.	22.55%.

D.

Action of Bromine upon Paratoluquinoline in the Presence of Fuming Sulphuric Acid.

Ten grams of paratoluquinoline sulphate were dissolved in fifty grams of fuming sulphuric acid, after which twenty grams of bromine were slowly added, the mixture being in the meantime kept cool with ice. After heating the mixture four hours on the water bath, it was poured into ice cold sulphurous acid and made alkaline with ammonia. The white precipitate which formed was filtered off and washed. The product crystallizes from acetic acid in colorless needles which melt at 135° -136°. The yield is almost quantitative.

Bromine Determination.

.1569 grms. substance gave .1947 grms. AgBr.

Computed:	Found:
$C_{10}H_7NBr_2$,	
Br=53.15%.	52.81%.

Platinum Double Salt.

Platinum chloride precipitates the crystalline double salt from the HCl solution of the base.

Platinum Determination.

.1761 grms. substance, dried at 100°, gave by ignition .0340 grms. platinum.

Computed:	Found:
$(C_{10}H_7NBr_2)_2H_2PtCl_6$,	
Pt=19.27%.	19.31%.

E.

Action of Iodine upon Paratoluquinoline in the Presence of Fuming Sulphuric Acid.

Twenty grams of powdered iodine were mixed with ten grams of paratoluquinoline sulphate. The mixture was then added in small quantities to fifty grams of fuming sulphuric acid. After five hours heating on the water bath, the resulting product was poured into ice cold sulphurous acid. The impure yellow product thus obtained was filtered, washed with hot water until the yellow color had disappeared, and recrystallized from alcohol. The di-iodoparatoluquinoline thus obtained forms silky white needles, which melt at 135—136°. Yield 72%.

Iodine Determination.

.1562 grms. substance gave .1858 grms. AgI.

Computed:	Found:
$C_{10}H_7NI_2$,	
I=64.05%.	64.27%.

Nitrogen Determination.

.2601 grms. substance gave 9.3 c. c. N. at 730 m. m. and 21°.

Computed:	Found:
N=3.54%.	3.90%.

Platinum Double Salt.

After adding platinum chloride to the HCl solution of the base, yellow crystals of the platinum double salt crystallize out.

Platinum Determination.

.1576 grms. substance gave .0253 grms. platinum.

Computed:	Found:
$(C_{10}H_7NI_2)_2 H_2 Pt Cl_6$,	
Pt=16.24%.	16.05%.

Nitro-iodo-paratoluquinoline.

Five grams di-iodo-paratoluquinoline were heated with an excess of fuming nitric acid for two hours. On pouring the resulting solution into water a precipitate of nitro-iodo-paratoluquinoline forms, which crystallizes in yellow needles from acetic acid. Melting point 133°.

Iodine Determination.

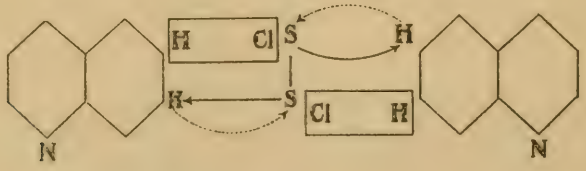
.1374 grms. substance gave .1034 grms. AgI.

Computed:	Found:
$C_{10}H_7NINO_2$,	
I=40.44%.	40.69%.

In the preceding investigations it has been made clear that the formerly accepted formulas for thiochinanthrene, its isomer, and its homologues must be changed in such a manner that they shall include tetravalent and not divalent sulphur as heretofore supposed. The valences are satisfied in that the sulphur atoms are bound together

by one bond, two bonds are attached to carbons of separate quinoline molecules, and the fourth bond is satisfied through a hydrogen atom. This hydrogen atom is not replaceable by acetyl or bromine; the sulphur has however the power of becoming hexavalent and adding two atoms of bromine, a molecule of nitric acid, and in combination with the second sulphur atom a molecule of sulphuric acid. A similar phenomenon was observed by Saytzeff[1] and Beckman[2] in the case of sulphoxides. These substances can also add a molecule of nitric acid, thus one can obtain, for instance, the compound $(CH_3)_2 SO \cdot HNO_3$. Colby and McLaughlin[3] obtained when nitrating $(C_6H_5)_2 SO$, besides two nitro bodies, a by-product which was very similar to the above mentioned dimethylsulphoxide nitrate, and which probably was a diphenylsulphoxide nitrate.

Paratoluthiochinanthrene shows analagous properties to the first of the series, and like it is not sensitive to oxidizing or reducing agents; SCl_2 by the reaction gives, as with quinoline, besides the sulphur base, chlorine substitution products. A monochlorine body was in this case however not obtained. Here di and trichlor bodies were formed. The reaction appears in a much clearer light than before. A molecule of S_2Cl_2 acts upon two molecules of quinoline in the following manner: First, two atoms of hydrogen, one from each quinoline molecule, unite with two atoms of chlorine from the chloride, forming hydrochloric acid; the neighboring hydrogen atoms in the quinoline molecules then change over to the sulphur atoms, leaving each a carbon valence to be satisfied with the fourth sulphur valence.



(1) Liebigs Annalen der Chemie, 144, 148.
(2) Journal Prakt. Chemie, 17, 471.
(3) Berliner Berichte, 20, 198.

When SCl_2 is used this breaks down into S_2Cl_2 and Cl_2, according to the formula $2\,SCl_2 = S_2Cl_2 + 2\,Cl$, thus furnishing the necessary chlorine for the chlorination process.

Lastly, fuming sulphuric acid has again been shown to be a splendid aid in the substitution of bromine and iodine in quinolines.

Volume I Number 3

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STATE HIGHWAY SYSTEMS

By John B. Phillips

Until recent years the energy devoted to improving transportation facilities has been largely expended in building railroads and improving waterways. The common roads have been neglected. This is especially true in the United States. But the fall in the price of agricultural products and the fact that these products must be hauled for considerable distances over common roads before reaching the market, has made the farmers clamorous for a better highway system. This demand for road improvement has been intensified by the establishment of rural mail service and the use of motor vehicles.

In the United States the construction of highways has generally been left to the farmers owning lands adjoining the road. The tax assessed for highway purposes is commuted to labor and worked out. The roads are divided into strips of varying lengths called districts. The residents of each district annually elect one of their number to act as overseer. At a time when there is a slackness of work on the farm the overseer assembles the farmers for work on the road. The whole proceeding is in the nature of a social gathering, the hours of labor are short and the allowance in highway tax paid is large.

The conspicuous failure of this system has recently led to new legislation in the endeavor to settle the problem. Thus far the most successful method has been that in which the state undertakes the work of road improvement, the contract being let and the construction supervised by a state officer. The expense is usually apportioned among the state, county and local division. This system with some modifications has recently been adopted in New York, Massachusetts, Connecticut, New Jersey and Vermont. California, Maine and North Carolina have rudimentary state road systems and in Cal-

ifornia a constitutional amendment providing for a state system somewhat similar to that of Massachusetts was adopted in November, 1902.

Below are digests of the laws of those states which provide state aid or supervision in the improvement of highways:

Massachusetts.

In Massachusetts state road improvement is under the control of a commission of three members appointed by the governor; term, three years; salary, president, $3,500; others, $2,500. On petition of the county commissioners, mayor and aldermen of a city or selectmen of a town requesting state improvement of any highway, the commission investigate and determine the necessity for improvement. If the commission act favorably on the petition, the road becomes a state road and remains permanently under the commission's control.

When about to construct any highway, the commission is required to give notice to cities and towns through which the road passes, and may contract with them without advertisement for its construction. If not contracted for by cities and towns, the work is let in the usual manner to private parties.

Construction of state roads must be fairly apportioned among the several counties, and not more than ten miles may be built in any one county in one year except by written consent of the governor and council.

One fourth of the expense of highway improvement in any county with interest at three per cent. must be repaid by the county to the state within six years as the commission and state auditor may determine, taking into consideration the financial condition of the county.[1]

Annual expenditure for repairs to the amount of $50 a mile is charged to the towns and cities where made. The tax thus collected is turned over to the commission to be used for road improvement.

(1) Mass., '95, Chap. 347.

To meet the expense of improving roads, a state loan has been made and a sinking fund provided. Five hundred thousand dollars was appropriated annually for 1900 and 1901.

Five per cent. of the annual appropriations for road improvement is to be spent in amounts at the discretion of the commission, in towns where no state road has been built. Such improvement is to be made only on petition of the selectmen and when made the road remains a town highway.[1]

New Jersey.

In New Jersey the improvement of roads is in the hands of the county boards of chosen freeholders, acting under the supervision of a state road commissioner appointed by the governor for three years at a salary of $2,500. Whenever the board of chosen freeholders determine to improve any road in the county, estimates, plans and specifications are filed with the state road commissioner. If after investigation, the commissioner decides that the improvement is necessary and can be made within the state appropriation, he may approve the plans. The director of chosen freeholders then advertises for bids and lets the contract, and the state commisioner appoints a supervisor to oversee construction. On completion of work the supervisor must file itemized statements of cost with the board of chosen freeholders and state commissioner. One-third of the cost of improvement is paid by the state and two-thirds by the county. The annual state appropriation is $150,000.[2] When completed, the road is a county road and must be maintained by the board of chosen freeholders. They must appoint a road supervisor and fix his salary.[3]

Proceedings for the improvement of a road may also be begun by individuals. On petition of owners of two-thirds of property in lineal feet or area abutting on a highway, stating that they will pay 10 per cent. of the cost of improvement, the board of chosen free-

(1) Mass., '00, Chap. 432.
(2) N. J., '99, Chap. 43.
(3) N. J., '95, Chap. 443.

holders must cause the improvement to be made in the same manner as above described. When completed, the circuit court appoints commissioners to assess the benefits on abutting owners.[1]

Connecticut.

Road improvement in Connecticut is under the supervision of a state highway commissioner appointed by the governor for four years at a salary of $3,000. The commissioner makes a biennial report to the legislature. Whenever a town determines to improve a road, the selectmen, with the approval of the state commissioner select the portion to be improved. The selectmen cause a survey to be made and submit it to the commissioner, who prepares plans and specifications and estimates the cost. If the cost is not to exceed $1,000, the commissioner may allow the town to do the work without competition; otherwise, the selectmen must advertise and let the contract to the lowest bidder. Contracts must be approved by the state commissioner and filed in his office. The highway commissioner may appoint inspectors to supervise construction, and fix their salaries which are paid by the state but not more than $10,000 may be so spent annually. When completed the road is kept in repair by the town.[2]

In towns of more than $1,000,000 assessed valuation, two-thirds and in other towns three-fourths of the cost of road improvement is paid by the state. The balance is paid by the towns. In one year, not more than $4,500 of state money may be spent in any one town. Certificates of cost of improvement must be filed with the state highway commissioner. No money may be paid out by the state controller for road purposes except on certificate of the state highway commissioner. Total state annual payments may not exceed $225,000.

New York.

The New York system of state aid and control in the improvement of highways was adopted in 1898. The initative is taken by

(1) N. J., '95, Chap. 223; '99, Chap. 44.
(2) Ct., '99, Chap. 175; '01, Chap. 149.

a majority of the property owners facing the road to be improved or by the township and is in the form of a petition to the board of supervisors. If the petition is from the abutting property owners, the board must apply to the state engineer for aid in the improvement. Petitions from townships are not mandatory on the board of supervisors. The application to the state engineer must designate and describe the road to be improved and give its length. The state engineer then examines the road and if he considers it of sufficient importance to warrant state aid, he orders a survey, together with plans and estimates of the cost of the improvement. These are then submitted by him to the board of supervisors, and if approved by the board, the contract is let and the road constructed under the supervision of the state engineer.

Fifty per cent. of the cost of the improvement is paid by the state, thirty-five per cent. by the county, and fifteen per cent. by the town, or if the property owners have petitioned, by those whose lands are benefited. When completed the road must be kept in repair by the town.

The state engineer is required to collect statistics and information concerning roads and advise with local officers and persons interested in road improvement. He must hold at least one public meeting annually in each county in the interest of good roads.[1]

Since the passage of the law the state appropriations in aid of highways have been as follows: 1898, $50,000; 1899, $50,000; 1900, $150,000; 1901, $420,000.

Maine.

In this state there is no state commissioner of highways, but state aid is given to towns constructing improved roads. On the request of the municipal officers of any town, the county commissioners designate some road as a main thoroughfare, and the road thus designated becomes a state road.

The town builds the road and the work is inspected and approved by the county commissioners. The commissioners certify to

(1) N. Y., '98, Chap. 115.

the governor and council that the road is completed, and state the amount that has been expended by the town. The town is entitled to receive from the state an amount equal to that expended in road improvement up to $100, this being the maximum amount of state aid that can be given any one town. Towns must apply for state aid to the secretary of state. Applications are filed in the order in which received. If the appropriation is insufficient to pay all claims in any one year, the last claims received are paid from the following year's appropriation. The appropriation for 1902 was $15,000.[1]

Vermont.

In this state there is a state highway commissioner appointed by the governor for a term of two years; compensation, $4 a day and traveling expenses. The commissioner supervises the expenditure by towns of state highway money and gives advice to town road commissioners.

A state tax of one-half mill on a dollar of valuation is annually levied and distributed to the towns in proportion to their road mileage.[2]

The town road commissioners make to the state commissioner a detailed report of roads built and the expense of building. If the state commissioner is satisfied that the town has spent an amount equal to its portion of state highway tax, he issues his certificate on the state treasurer in favor of the town.

The road commissioners of the various towns in a county constitute a county board of road commissioners. The county boards meet annually in their respective counties with the state commissioner. The state commissioner may employ experts to instruct in road maintenance and building, and he may personally direct the work in towns.[3]

North Carolina.

In North Carolina there is a state highway commission composed of the commissioner of agriculture and state geologist. No

(1) Me., '01, Chap. 285.
(2) Vt. Statutes, '94, §3434.
(3) Vt., '98, Chap. 65.

additional compensation is allowed these officers for services as highway commissioners. The commission is to advise with county and town authorities concerning road and bridge improvements and may furnish without charge to the local authorities the services of an engineer to aid in road building. The commission also makes rules and regulations for the employment of convicts on roads, and issues bulletins. There is no state highway system or financial aid to localities.[1]

California.

There is a department of highways in California under the control of a commissioner appointed by the governor for a term of four years at a salary of $3,000. The department is to take possession of the highways that have been or may be declared state roads and have charge of the state expenditures for highway purposes. It is also required to advise with local road authorities, investigate road construction, issue bulletins, and report biennially to the governor. A number of roads specially built by the state are under the control of the commission and a constitutional amendment empowering the legislature to establish a state highway system and aid in the construction of local roads was adopted in November, 1902.[2]

(1) N. C., '01, Chap. 50.
(2) Cal., '97, Chap. 267; '01, p. 960.

THE FOURTEENTH AMENDMENT

By Charles E. Chadsey

The history of the formulation of the fourteenth amendment to the Constitution of the United States is interesting as an example of the workings of Congress during the struggle over the reconstruction of the Southern States after the Civil War. The subjection of these states in the spring of 1865 made necessary some plan for their reorganization. The history of the development of the theory of reconstruction, a development which had been in process since the beginning of the war and which was crystallized by the active opposition of President Johnson, bears indirectly upon the passage of the resolution submitting the amendment.

The original attitude of Congress towards the Southern States was embodied in the restoration idea. It was thought that when the war was over the states would return to their old allegiance and that the relations which formerly existed between them and the central government would be restored. As the war dragged on, more radical feelings came to predominate and the majority came to believe that the South when crushed should be considered as subjugated territory entirely at the mercy of the central government. The death of Lincoln and the accession of Johnson who speedily antagonized this radical element resulted in the drawing of party lines with great strictness.

As was only natural, the Republican party possessed an overwhelming majority in Congress. A large proportion of these Republicans thoroughly distrusted the South. It seemed to them that a people who for four years had been using every energy in the effort to destroy the Union could not possibly be serious in their claim that they now proposed to be good, peaceable and loyal citizens. In their eyes they had proven themselves traitors and

traitors they must be considered until time could in part destroy the keenness of their recollections. Anything which approached clemency was to be suspiciously watched as possible weakness or insincerity. It was this feeling of intense suspicion and distrust which caused these radical Republicans to look upon Johnson with decided disfavor while he was so rapidly carrying out his policy of the restoration of the state governments during 1865.

However, long before it was definitely decided just what policy should be adopted in the reconstruction of the South, it was recognized that sooner or later there would be legally organized State governments in the conquered districts and that these states would be entitled to representation in Congress. As the Constitution then stood, there would be nothing to prevent these states from legally reversing all their actions after they had successfully passed through the preliminary stages, which the forthcoming reconstruction plans might require. Therefore good politics demanded that the Constitution be amended so as to prevent the most serious of the dangers which they believed threatened them.

There already had grown up throughout the North a strong feeling in favor of giving the negro the ballot. It is an open question as to how far humanitarian ideas are to be held responsible for this feeling. It is to be feared that political plans were largely responsible. The Republicans had good reason to believe that they could control almost the entire negro vote as the freedmen could easily be persuaded that their present freedom was due entirely to the Republicans and that the permanence of their freedom as well as their future prosperity could be assured only as a result of their unswerving loyalty to the party. Under these conditions they figured that the enfranchisement of the negro, especially if this could be coupled with some restrictions on the active participants of the Rebellion, would insure for an indefinite period Republican control in the Southern States.

On the other hand to give the negro the ballot entailed no damage or hardship to the Northern States for the vote there would be practically infinitesimal. The plan therefore possessed everything

to commend and nothing to condemn. That there was back of and in addition to this a strong sentiment justifying the enfranchisement there can be no doubt but it seems probable that this sentiment was used to justify any further congressional plans rather than that the congressional demand arose in response to the sentiment.

With a Congress determined to enforce negro suffrage upon the South, the easiest and most natural plan would have been to submit the amendment which afterwards became the fifteenth amendment. But Congress was not quite ready for so radical a step. There were many who were willing to secure if possible Republican predominance in the South, who were not satisfied that the negro vote would remain inconsiderable in all the Northern States. Again public sentiment in the North could not as yet be safely counted on to endorse at the polls such an amendment. There was found to be a considerable feeling existing which demanded that the privileges of statehood should not be infringed upon by a measure which practically took away from a state the right to regulate the elective franchise as it should see fit. These considerations made some less radical step advisable.

The first steps looking toward a solution of the problem were taken by the committee on reconstruction, a special committee which had been authorized to have referred to it all matters relating to the organization of the Southern States. Mr. Thaddeus Stevens of Pennsylvania, a member of this committee and the leader of the extreme radical wing of the House of Representatives reported a joint resolution on January 22nd, 1866, which provided "that whenever the elective franchise shall be denied or abridged in any state on account of race or color, all persons of such race or color shall be excluded from the basis of representation."

This resolution although couched in general terms was obviously drawn up with the colored race in mind. Mr. Stevens in speaking in its support attempted to defend it on general grounds. While a state would be at liberty to fix whatever qualifications for the franchise which it might see fit, this resolution would say: "If you exclude from the right of suffrage Frenchmen, Irishmen, or any

particular class of people, none of that class of people shall be counted on in fixing your representation in this House." The natural effect of such a clause would be that if any of the Southern States should be unwilling to trust her governmental machinery to the mercy of the horde of newly-emancipated blacks the vast majority of whom did not have the slightest conception of the responsibilities of citizenship, there would be no alternative but a lessening of her representation in Congress by nearly or quite one-half. The aggravation of such a condition was not lessened by another condition which contained practically the existing arrangement, apportioning direct taxes among the states according to the entire population whether any were excluded from representation or not.

With such a resolution adopted as part of the Constitution, two alternatives would have been open to the South, a grossly disproportionately small representation, or their governments thrown open to the colored man who would in turn be the tool of the unprincipled white adventurer. Conservative members of Congress were not slow to point out this obvious fact. Thus Chandler of New York in the debate on the resolution said that if it were to go into effect "you give the control of the ballot box to the negro who will hereafter by this system of enactments become the majority of the people under the democratic and established law of our whole policy and the constitution and we must bow to the will of the people. Ingraft the black man into the term "people" and you surrender the South to the black race and the question comes up not between slave and free but between black and white."

It was impossible at this time to foresee how successfully the South would be able to use terrorism to prevent black domination. The systematic oppression which has been practiced has alone prevented the realization of the prophecy which the carpet-bag government at one time threatened to fulfill. It is yet too soon for judicial opinion to be expressed upon the ethical and political principles involved in the system adopted by the South as a last resort to prevent what they honestly felt was ruining their governments. The passage of the Fourteenth Amendment forced the issue upon them and

judgment upon the South will necessarily include judgment upon the wisdom of the amendment.

After extended debate, the resolution was adopted by the House on January 31st, 1866, by a vote of 120 to 46. The crucial test however was to come in the Senate which began its consideration of the resolution on February 5th. As not infrequently happens in questions where there is a great variance of opinion, there was vigorous opposition both from the extremely radical and conservative wings and between the two the resolution fared hardly. There was one faction which would be satisfied with nothing less than unconditional granting of the suffrage. They were not inclined to compromise on anything less radical. In their estimation a vital principle was at stake. The corner-stone of our whole governmental structure was based upon the principles of liberty and equality. Anything which temporized, which failed to grant to the negro the complete civil and political equality which the white possessed endangered this corner-stone. The proposition that this was a white man's government was violently repudiated as a miserable makeshift for oppression.

Sumner pleaded in opposition that "it is not enough that you have given Liberty. By the same title that we claim Liberty do we claim Equality also. One cannot be denied without the other. * * * They are the two vital principles of a Republican government without which a government although Republican in name cannot be Republican in fact." The pages of the Congressional Globe are full of sentiments of a similar nature. This faction occupied a great vantage ground. Of the loftiness of the sentiments expressed and of the sincerity of Sumner and his followers there could be no doubt. The terrible dangers of the Civil War had left their impress on these men and the noble ideas which proved the watchwords to victory after the Emancipation Proclamation still permeated the discussions in the congressional halls. Yet never was the fact that tearing down social structures is vastly easier than building new ones better demonstrated than in these very debates. It has become a truism to say that liberty is too often interpreted by the ignorant

to mean license and it is scarcely less evident that the principle of political equality is liable to equal perversion when given unreservedly to those who are absolutely unfitted for its exercise. Noble principles too often cast a roseate hue over stern conditions and the impulses of those under their influence are apt to sacrifice present necessities in the hope of the realization of grand ideals.

In contrast to these lofty ideals which were put forward as the strongest arguments against the resolution, the Conservatives opposed it by arguing the great advantage of a speedy re-adjustment to old conditions. If the South were forced against its will to give the negroes the ballot there would inevitably result a bitterness which time would be slow to wipe out, which in fact could only be blotted out when the negro had been raised to the point where he would be able to cast the ballot intelligently. It would take generations to accomplish this if it could be accomplished at all, while in the meantime the South would be exposed to the numberless dangers if not utter ruin which might accompany the free ballot. The states of right should determine the qualifications for the elective franchise themselves and when the negroes could be safely entrusted with self-government, the privilege should come voluntarily from the state. Until then it was boldly urged by some that our government should continue to be what it had been in the past—a white man's government.

During the debate, numerous amendments to the resolution were offered, among them being one submitted by Senator Howard of Michigan, which is deserving of special attention. His plan would have given the ballot to the colored man under certain restrictions. All over twenty-one years of age who had been enlisted in the army or navy were to have the right to vote without further qualifications. On all others a property or an educational qualification was to be imposed. The ability to read and write or the possession of property valued at $250 or more was the qualification required. This plan is substantially that recommended by President Johnson in a telegram to the governor of Mississippi, and certainly contained much that can now be heartily commended.[1] If such a

[1] McPherson, History of the Reconstruction, p. 19.

measure could have been carried, the most perplexing features of the race problem would have been avoided and the era of carpet-bag government and the KuKlux Klan would have been impossibilities. The plan, however, met with but little favor, and the reason for this is not hard to see. In the mind of the average Republican, good citizenship and loyalty to the government could be possessed only by those who were identified with the Republican party. This resolution would leave the government of the Southern States in the hands of the Southern Whites who had almost to a man participated in the Rebellion, and who were all Democrats as a matter of course. Under these conditions they felt that the passage of Senator Howard's resolution would be an open invitation for the confederate leaders to resume their control of the South, and to speedily introduce a social system little if any different from the old slavery.

The debate was closed on March 9th. No amendment to the original resolution proved satisfactory and the resolution itself failed to receive the requisite two-thirds majority, the vote standing twenty-five yeas to twenty-two nays. The failure of the resolution was a great disappointment to many, but before two months had passed by, the joint committee had framed another joint resolution which it was hoped would meet with greater favor [1] Mr. Stevens, like Senator Sumner, would have preferred far more radical measures but as he said in his opening speech supporting it, he thought it was all that public opinion would justify.

The section which corresponded in its general purport to the resolution submitted in January was supposed to be worded rather more conservatively. It read as reported from the committee that "Representatives shall be apportioned among the states according to their respective numbers, counting the whole number of persons, excluding Indians not taxed. But whenever in any state the elective franchise shall be denied to any male citizen not less than twenty-one years of age or in any way abridged except for participation in rebellion or other crime, the basis of representation in such states shall be reduced in the proportion which the number of male citizens

(1) Congressional Globe, 39th Congress, 1st Session, p. 2286.

shall bear to the whole number of male citizens not less than twenty-one years of age."

This section can easily be seen to be far less radical than the original resolution. According to it if a very small proportion or as Stevens more strongly put the case, "If one of the injured race was excluded, the state should forfeit the right to have any of them represented." But this modified section would permit a considerable abridgement of the suffrage without any serious loss in representation. A certain amount of discrimination could be indulged in safely. Stevens, in fact, was very bitter towards those who had caused the failure of the first resolution in the Senate. He declared the defeat due to the "united forces of self-righteous Republicans and unrighteous Copperheads * * * the large stride which we in vain proposed is dead; the murderers must answer to the suffering race * * * A load of misery must sit heavily upon their souls." * * * [1]

As was expected, the modified form of this section proved less obnoxious to some and although many felt that it was thoroughly unsatisfactory, it passed both Houses in the slightly amended form in which it now stands.

The first section of the resolution met with but little opposition although the form was modified and some additional clauses were inserted before it was finally adopted. The Civil Rights Bill was designed to afford a plan by which the central government could carry out its obligations to guarantee to the freedman the privileges and immunities of citizenship. This section simply incorporated in the Constitution the general principle that all persons born or naturalized in the United States were to have unabridged all that citizenship involved.

There was one proposition in the original resolution in which the prejudice and fear of the radical Northerner cropped out most emphatically. It proposed that "until the fourth day of July, in the year 1870, all persons who voluntarily adhered to the late insurrection should be excluded from the right to vote for Representatives and for President and Vice-President of the United States."

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[1] Congressional Globe, 39th Congress, 1st Session, pp. 2459-60.

The old fear that the ex-confederate would immediately manipulate the newly reorganized States to his own interests accounts for this proposition. Even in the House there was considerable dissatisfaction manifested by many who otherwise favored the resolution. It was felt that it worked an injustice to many and that it was inspired by feelings of sectional hate. Some did not hesitate to call it an action directly opposed to the original spirit of the Constitution and a decided step toward centralization. Mr. Raymond of New York who supported the amendment with the exception of this section, voiced the sentiments of many when he said that "if history teaches anything, if any principle is established by the concurrent annals of all nations, it is that sentiment cannot be coerced; that opinions even cannot be controlled by force; and that with any people fit to be free, or fit to be the countrymen of men who are free, all such efforts defeat themselves and intensify and perpetuate the hostilities sought to be overcome."

The sentiments of those favoring the section were expressed by statements such as McKee of Kentucky made when he declared that without this "the widows and orphans of the slain soldiers of the Republic" would be insulted by having traitors to frame their laws. "The loyal alone should rule the country which they alone had saved." Mr. Stevens in closing the debate in the House, was most urgent in his plea that this section should not be stricken out. In his usual extravagant but unquestionably sincere manner, he denounced the confederates as unfit for the present to wield the ballot. He did not wish to "sit side by side with men whose garments smell of the blood of my kindred."

It was soon evident that something more conservative would be substituted in the Senate. Senator Wade of Ohio proposed that the only discrimination which should be made which would affect representation "should be in virtue of impartial qualifications founded on intelligence or property or because of alienage or for participation in rebellion or other crimes. This was immediately attacked on the ground that property should never be made a qualification for voting. Senator Wilson proposed as a substitute "that no person who has

resigned or abandoned or may resign or abandon any office under the United States, and has taken or may take part in rebellion against the government thereof, shall be eligible to any office under the United States or of any state." Senator Clark of New Hampshire proposed that the section be worded to the effect that no one who after "having taken an oath to support the Constitution should have engaged in the Rebellion" should be eligible to any office under the government. These propositions of Wilson and Clark met with considerable favor as they seemed to embody the ideas upon which agreement could be most easily reached.

The whole debate had shown that there was no faction in the Senate determined to force this third section in its original form. On the 29th the motion striking out the section was carried unanimously. On the same day Senator Howard as the spokesman of the Republican caucus presented a series of amendments to the original joint resolution. The third amendment in this series substituted for the section stricken out the third section as it now stands. This was practically a combination of the resolutions submitted by Senators Wilson and Clark with the additional saving clause that Congress might remove the disability by a two-thirds vote. The conservative minority made a vigorous fight against the adoption of these resolutions, dilatory tactics being resorted to, amendment after amendment being proposed and promptly rejected. The caucus agreement was rigidly adhered to and the substituted section was agreed to on the following day.

The fourth section of the Joint Resolution read as follows: "Neither the United States nor any state shall assume or pay any debt or obligation already incurred, or which may be incurred in aid of insurrection against the United States, or any claim for compensation for loss of involuntary service." This section excited but little opposition. It passed the House unchanged, but in the Senate quite material additions and modifications were made. On June 4th, on Senator Howard's motion, there was inserted in the section the principle that "The obligations of the United States in suppressing insurrection or in defense of the Union or for payment of bounties or pen-

sions incident thereto shall remain inviolate." On June 8th Senator Clark moved the substitution of the section as it now stands. This was merely changing the phraseology and the motion was agreed to without opposition.

The resolution as amended by the Senate was taken up for consideration by the House on June 13th, and the amendments were concurred in after a short debate. The resolution was then ready to go before the states for ratification as the Fourteenth Amendment.

Interest was centered on the action which should be taken by the insurrectionary states. The North was not long left in doubt. Texas, Georgia and Florida were the first to consider the amendment. All promptly rejected it and within a few weeks the entire South including the three loyal states of Delaware, Maryland and Kentucky had refused to ratify.[1] No one could be surprised at this result. The negro population in all of these states was very large. The sentiment against granting the franchise was extremely strong while the penalty of decreased representation in case of discrimination against the negro was exceedingly distasteful. The disabling clause, affecting as it did a great many of the most prominent Southerners, was very unpopular and was widely felt to be unjust. A large majority of all the legislatures felt that it would be little short of odious for them to vote in favor of such a proposition. The provision practically requiring them to repudiate their debts would of itself have caused violent opposition. Many claimed that so far as possible they should be permitted to pay their debts without interference. These and other reasons, combined with the natural bitterness already existing towards Congress which was only intensified by its attitude during the debates then in progress over the Reconstruction Bill, made any other result impossible.

The rejection in turn reflected on the attitude of Congress toward the South, an attitude which found expression in Section 5 of the first Reconstruction Bill. It was there declared among the precedents necessary to the reconstruction of a state and the admission of its representatives to Congress that the state should ratify the

(1) McPherson, History of the Reconstruction, p. 194.

Fourteenth Amendment. To offset the known influence which those included in the disabling clause had had in the Southern legislatures rejecting the amendment, a proviso was added to the section excluding them from membership in the Constitutional Conventions to be convened in pursuance to the act.

The passage of this act on March 2nd, 1867, insured the victory of Congress and the enforced ratification of the amendment. From the strictly legal point of view, it might well be doubted whether such a ratification could be held to be constitutional, but the practical experience of following years convinced the South that they held in their hands an instrument which would neutralize the effects of the amendment. Terrorism proved to be the logical sequel to enforced ratification.

Even with the impetus which the Reconstruction Act was supposed to give to the adoption of the amendment, the ratifications dragged on slowly. Ohio and New Jersey had ratified the amendment but later reconsidered their action and repealed their ratification resolutions. For a time it looked as if their action would seriously complicate matters. Secretary Seward on July 20th, 1868, issued a proclamation concerning the amendment in which he showed plainly his doubt as to the effect of their action.[1] If this ratification was to be considered legal in spite of their later repeal measures, the requisite two-thirds majority had already been secured and the amendment could be considered in effect. Otherwise a final proclamation would have to await further action from dilatory states. The large working Republican majority of Congress settled the question on the following day by passing a resolution affirming that the first resolutions of these states could not be repealed.

Seward accepted this interpretation and one week later, July 28th, 1868, he issued the proclamation which declared the Fourteenth Amendment an integral part of the Constitution.

(1) McPherson, History of the Reconstruction, p. 379.

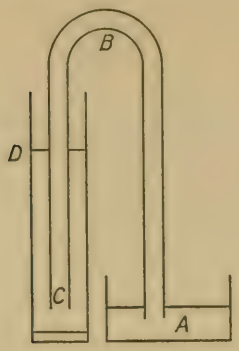
ON THE SIPHON

By William Duane

The writer wishes to call attention to an error that has crept into the text-books on general physics, written for high-school and university classes. Most of the books either state explicitly that a siphon will not work if the shorter of its two legs is longer than the column of liquid that would be supported by the air pressure, or else give explanations of the siphon, from which this follows as a legitimate conclusion. As a matter of fact, a siphon can be made to work and draw the liquid to a height considerably greater than that representing atmospheric pressure.

The writer usually illustrates this fact in his lectures by means of the following simple experiment: Let ABC in the figure be a glass siphon tube, both legs of which are 10 cm. or 15 cm. longer than the barometric column. The bore of the tube should be small (about $\frac{1}{10}$ sq. mm.) to work well. Let one of the legs, BC, dip down into a larger tube, CD, partly filled to D with mercury. Fill ABC with mercury, and start the siphon drawing mercury from C over to A in the usual way. In order to start the siphon the vertical height of B above the surface D of the mercury should be less than the length of a mercury barometer column, but as the flow continues, the mercury surface descends and keeps on descending until its vertical distance below C is considerably greater than this length.

To make this experiment work sufficiently well for demonstration purposes, excessive care in purifying the mercury and cleaning the glass is not necessary. Boiling the mercury in the actual tubes used, for instance, is superflous. With ordinary redistilled commercial mercury and tubes cleaned with alcohol the writer has made the siphon work to a height of seventy centimetres. As the altitude of the University laboratory, where the experiment was performed, is a



little over one mile, and the barometric pressure, therefore, only about sixty-one centimetres, this means that the siphon worked nine centimetres above the barometric height.

The most plausible explanation of the above fact is that the atmospheric pressure is not the only force pushing the mercury up the shorter leg. It is drawn up partly by the cohesive attraction of parts of the mercury for each other, and the column is kept from dwindling by the adhesive force exerted by the sides of the tube on the mercury.

It follows from the above that if a mercury siphon is placed under the receiver of an air pump, it can be made to work over a height of several centimetres, even though the air pressure is reduced to only a few millimetres. This experiment also has been shown to the writer's students. The apparatus was similar to that described above, except that the tubes were much shorter.

SOME SPECIAL ALGEBRAIC TRANSFORMATIONS REALIZED BY LINKAGES[1]

By Arnold Emch

1. In a paper on "Algebraic Transformations of a Complex Variable Realized by Linkages," recently published in the Transactions[2] of the American Mathematical Society, I have shown that *any number of algebraic relations between n complex variables may be realized by a plane linkage.*

I have considered in particular the algebraic relation

$$f(u, z)=0$$

between the complex variables u and z, which may be realized by a suitable combination of linkages for the relations

$$z=z_1+z_2,\ z=z_1z_2.$$

In conclusion linkages for these relations were described. Since the publication of the paper I have found that Professor P. Somoff, of Warsaw, in an article[3] on some applications of the kinematics of variable systems to linkages, described the same linkage which I have devised for $z=z_1+z_2$. The realization of $z=z_1z_2$ was based upon Kleiber's linkage (K)[4] as shown in Fig. 4 of my article in the "Transactions." The compound linkage however is not perfectly general. If the triangle $A_0A_2A_3$ is kept fixed, then the point A'_3 describes a circle and can, consequently, not reach every point of the Gaussian plane. In a recent letter to the writer, Professor Kleiber

(1) A part of this paper, with the title, "A Linkage for $u=e^{i\theta}z$ and Its Applications," has been presented to the Am. Math. Soc., Chicago, Sept., 1902.

(2) Vol. 3, No. 4, pp. 493-498, October, 1902.

(3) *Zeitschrift fur Mathematik und Physik*, Vol. 46, pp. 199-217, 1901.

(4) In what follows I shall designate Kleiber's linkage simply by K.

suggests how his linkage might be generalized in order to solve the general case $z=z_1z_2$. In this generalization my previous supperposition of two Kleiber linkages is not needed. In the following section a description of this new linkage, as it has been designed by the writer, shall be given.

2. Linkage for $z=z_1z_2$.

Consider in Fig. 1 the quadrilaterals $A_0C_1A_1D_1$ and $A_2C_3A_3D_3$, and based upon these construct two new linkages K_1 and K_2 simi-

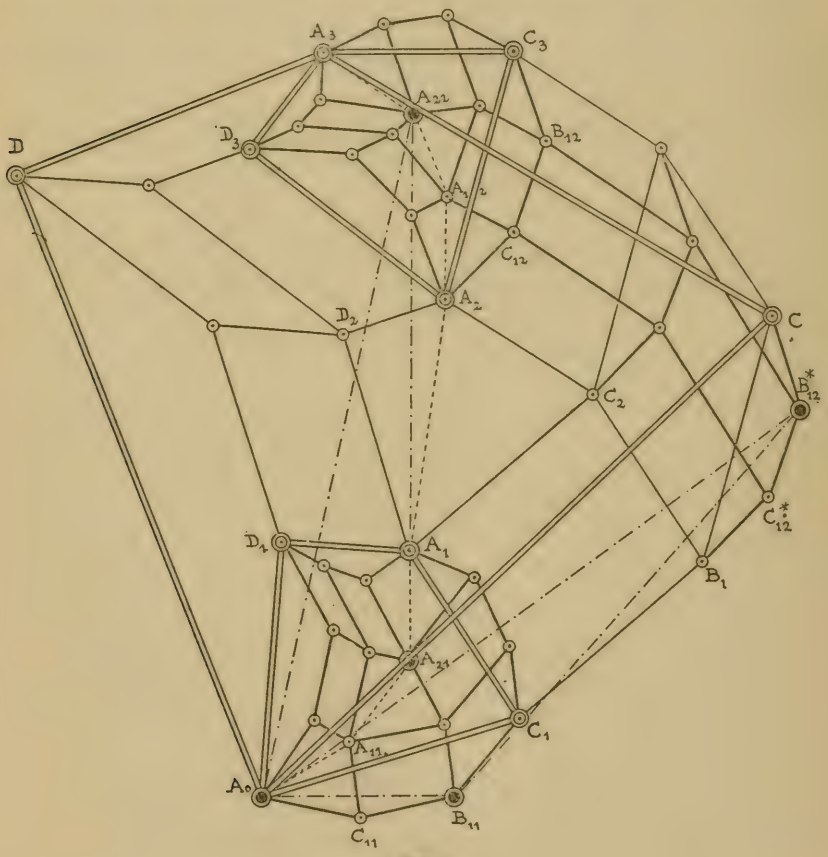


Fig. 1

lar to the linkage K which is based upon A_0CA_3D. If one of the above quadrilaterals is fixed then each of the remaining points of K_1 or K_2 has one degree of freedom left in its motion. Designate the points of K_1 and K_2 by the same letters as those of K and affix in K_1 everywhere the index 1, in K_2 the index 2. Then repeat the links $A_2C_{12}B_{12}C_3$, parallel and equal, on C_2B_2 and B_1C, and designate the pivots of the last links by $B_1C^*_{12}B^*_{12}C$. In this manner two similar polygonal lines

$$A_0C_{11}B_{11}C_1B_1C^*_{12}B^*_{12}C \backsim A_0A_{11}A_{21}A_1A_2A_{12}A_{22}A_3$$

are obtained. If the polygon $A_2C_3A_3D_3$ is fixed then A_{22} may describe a circle with A_3 as a center. We have now

$$\triangle A_0B_{11}B_{12} \backsim \triangle A_0A_{11}A_{22},$$

no matter how the whole linkage may be distorted. These triangles are otherwise independent of each other. Thus taking A_0A_{21} as the first real unit of the complex plane and designating the points B^*_{12}, A_{22} and B_{11} respectively by z, z_1 and z_2, we evidently have

$$z = z_1 z_2,$$

i. e., the realization of complex multiplication. The range of effectiveness of this, as of any other linkage is, of course, limited to a certain finite portion of the plane. This range, although in some cases small, always exists. (No interference between the parts of the linkage is allowed.)

By the linkage just described all the special cases may be obtained. Thus for $z_1 = z_2 = u$ we get $z = u^2$ and by inversion $u = \pm\sqrt{z}$.

In the following section I shall show, how the problem of many-valued functions may be solved in the case of the nth root. Putting z_2 = constant and $|z_2| = 1$ we have the rotation of the point z_1 through an angle = argument of z_2. For multiplication by a constant, Sylvester's pantograph may be used. Two years ago the writer has

constructed linkages for $z=u^n$, and the primitive inverse $u=z^{\frac{1}{n}}$, for the circular transformation $z=\frac{au+b}{cu+d}$, and finally a compound linkage for the general algebraic function, provided the linear factors were known. All this is consumed by my treatment of $f(u, z)=0$.

3. Linkage for $u=e^{i\theta}z$

The application of (2) to special cases is complicated. I have devised a simple linkage for rotation. Designating in the complex plane the points by the same letters as the complex numbers which they represent, the transformation

$$u=e^{i\theta}z$$

of z into u (θ=constant) represents a rotation of z through an angle θ; *i. e.*, $\overline{ou}=\overline{oz}$, $<uoz=\theta=$const.

Mechanically this rotation is accomplished by the linkage of Fig. 2. All links except AC=BD are equal. AC=BD may be assumed so that $<$AOC$=<$BOD$=\theta$. From the figure it follows that $<$AOB$=<$COD, hence also $<z$OB$=<u$OC and $<z$OC$=$

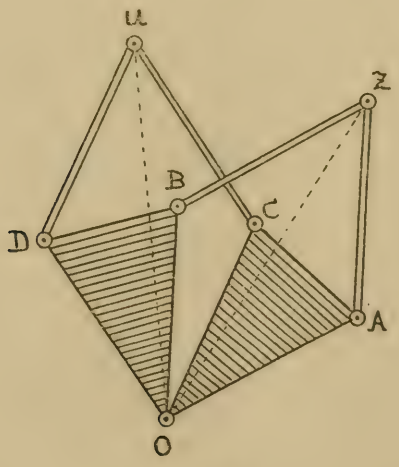


Fig. 2

$<uOB$. Now $<zOu<\pm 2<zOC+<COB$; but $<zOC=\pm<AOC\mp<zOA$ and $<COB=2<zOA-<AOC$, hence, on substitution,

$$<zOu=<AOC=\theta.$$

As the rhombes $OAzB$ and $OCuD$ are equal it follows that

$$\overline{ou}=\overline{oz}.$$

Thus, no matter how the linkage may be deformed and no matter how z may be assumed within its range, the point u represents $e^{i\theta}z$.

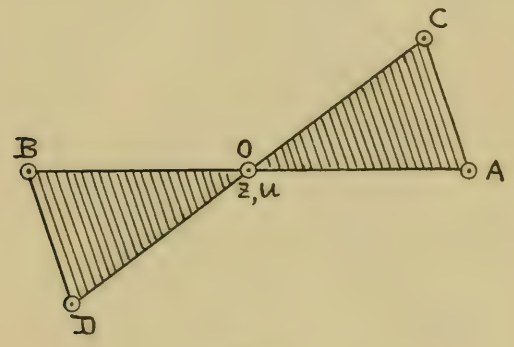


Fig. 3

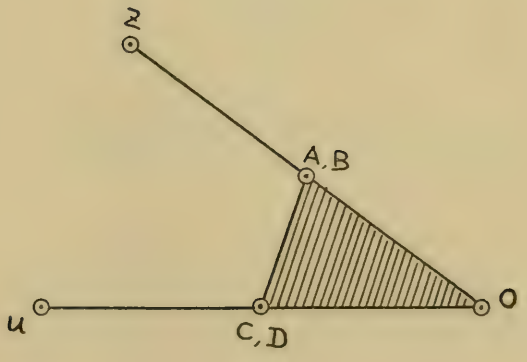


Fig. 4

By apparent deformations of Fig. 2, illustrated by Fig. 3 and 4, it is evident that the common range of z and u is a circle with 0 as a center and the length $\overline{2\ OA}$ as a radius.

Suppose now $\theta=\frac{2\pi}{n}$, where n is a positive integer and combine successively n equal rhombes in the same manner as $OCuD$ is connected to $OAzB$ in Fig. 2, and as it is illustrated in Fig. 5 in case

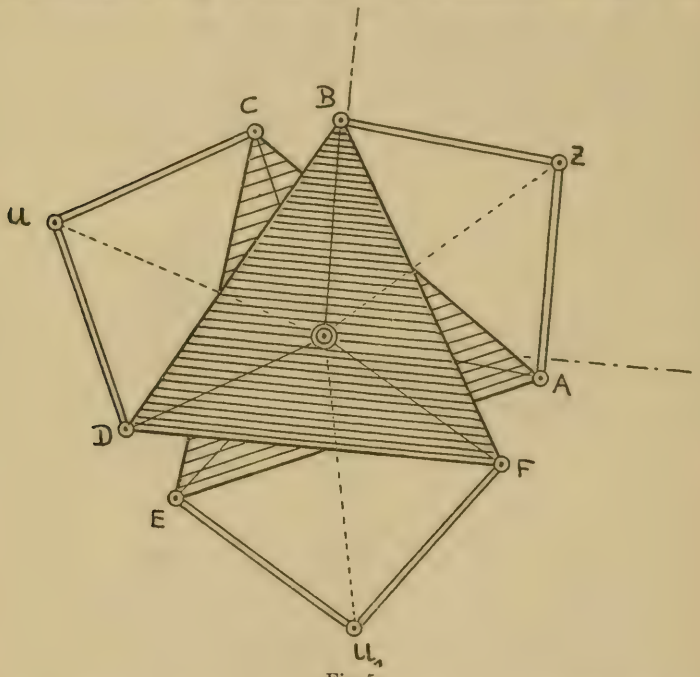


Fig. 5

of $n=3$. Thus a closed linkage is obtained in which the vertices $z, u, u_1, \ldots, u_{n-2}$ represent the complex numbers

$$e^{\frac{i2k\pi}{n}} \cdot z;\ k=0, 1, 2, \ldots, n-1$$

According to the general theorem it is possible to construct a linkage for:

$$w=z^u,$$

and consequently, by inversion, for the primitive n^{th} root of w, $z=\sqrt[n]{w}$. *If therefore in the compound linkage just described z represents this root, then the points u, u_1, . . . , u_{n-2} of the closed linkage will represent the other roots.*

4. Linkage for Special Collineations.

The linkage of Fig. 2 may be slightly generalized and then combined with a linkage of the same kind, as shown in Fig. 6. Take two rhombs AEPF and ACBD with the common joint, or pivot A; join E and C, and F and D by two equal links EC=FD so that $\angle EAC=FAD=\frac{\pi}{2}$. From plane geometry it follows easily $\angle EAF=\angle CAD$, *i. e.*, that the two rhombs are similar; further that PA⊥AB, no matter how the linkage may be distorted. *This linkage realizes*

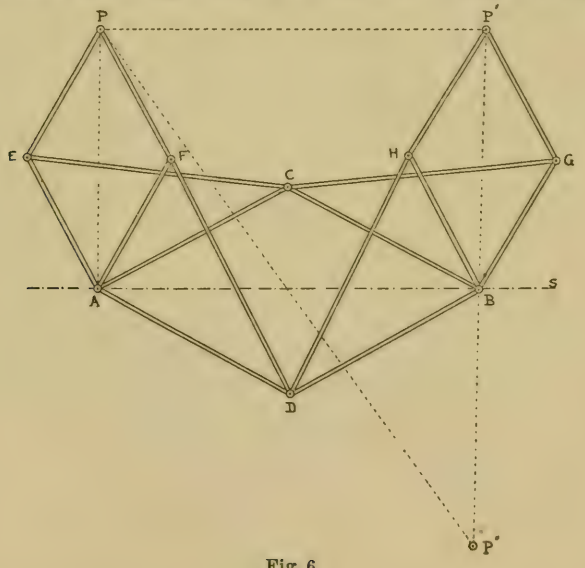


Fig. 6

therefore a variable right triangle PAB whose angles are constant. Joining in a symmetrical manner the rhomb BHP′G=AEPF to the previous linkage (CG=CE, HD=FD), *a variable rectangle $ABPP'$ is obtained whose sides have a constant ratio*, Fig. 6.

This linkage may be used to solve mechanically two interesting cases of collineations in a plane. If by two PEAUCELLIER INVERSORS the points A and B are forced to describe the same straight line s, in which an arbitrary point is taken as the origin of a Carterian co-ordinate-system, and s itself is assumed as the x-axis, we have, since $\frac{AB}{AP}=m$ (constant),

$$x'=x+my,$$
$$y'=y.$$

(x, y) and (x', y') designate the coördinates of P and P′ respectively. This collineation, whose group-property can easily be verified by the linkage, is called an *elation*, according to LIE,[1] and leaves areas invariant. The point of the line-element is infinitely distant in this case.

Instead of attaching the rhomb BHP′G as shown in the figure, it may be attached in the opposite direction so that the point P′ will now be at P′′. Determining the coördinates of P and P′′ as before, the point P′′ corresponds to P in an *oblique axial symmetry* with respect to s:

$$x''=x+my,$$
$$y''=-y.$$

Various combinations of the linkages described above, with pantographs, Koenig's perspectivograph,[2] etc., make it possible to realize in a practical form all collineations by linkages.[3] The possibility of this is proved by Koenig's general proposition.

JANUARY, 1903.

(1) *Vorlesungen über Continuierliche Gruppen*, p. 262, Teubner, Leipzig.
(2) *Comptes Rendus*, Vol. CXXXI, No. 26, p. 1179.
(3) This problem has been studied in detail by HERMANN EMCH, M. S., in his master's thesis —University of Colorado.

A PARTICULAR METHOD IN CENTROIDS

By J. J. Browne

The fundamental theorem of this article is, as will appear, well known and very elementary. It seems to have been used almost entirely in connection with problems on the *motion* of the Center of Mass. The idea of using it to determine centers of mass is not however new. The writer had his attention called to it when studying the subject, and its applicability in some elementary cases in Statics and Hydrostatics is obvious. He has not, however, after inquiry, been able to find that examples, as in this paper, of its use in connection with infinitesimals, have been published.

Almost all the results here established are usually arrived at by the Integral Calculus, and some of them require rather involved integrations.

The method seems interesting, though apparently rather limited in its application.

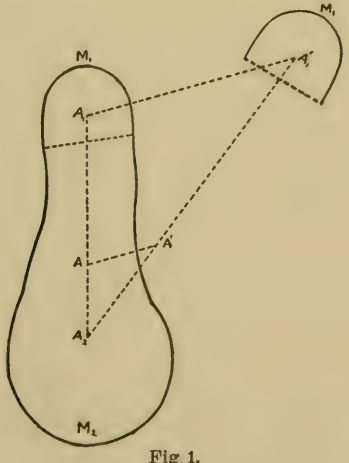


Fig 1.

1. THEOREM.

If a part of a body (or system) be moved, so that the Center of Mass of the part is transferred to a new positon, the Center of Mass of the body or system, still considered as a whole, is transferred in a parallel direction, and to a distance which is to the distance traveled by the Center of Mass of the part, as the mass of the part is to that of the whole.

Let the body represented in Fig. 1 be divided into two parts M_1 and M_2 by the dotted line, and let A_1 and A_2 be their respective Centers of Mass; then A the Center of Mass of the whole divides A_1A_2 so that $\frac{A_2A}{A_2A_1}=\frac{M_1}{M_1+M_2}$.

If now M_1 be transferred till its Center of Mass occupies the position A_1' we have if A′ be the new position of A

$$\frac{A_2A'}{A_2A_1'}=\frac{M_1}{M_1+M_2}$$

$\therefore \frac{A_2A}{A_2A_1}=\frac{A_2A'}{A_2A_1'}$ and $\therefore$ the triangles A_2AA' and $A_2A_1A_1'$ are similar.

$\therefore AA'$ is parallel to A_1A_1' and $\frac{AA'}{A_1A_1'}=\frac{M_1}{M_1+M_2}$ which proves the proposition.

2. *To find the center of mass of a uniform circular arc.*

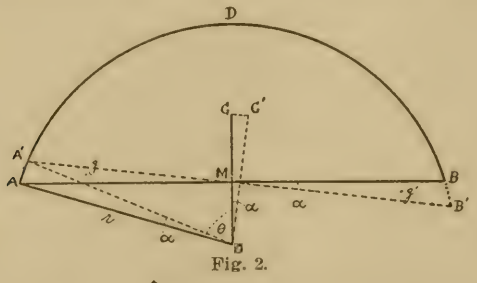


Fig. 2.

Let ADB (Fig. 2) be the arc, O the center of the circle and G the Center of Mass. OG is plainly perpendicular to AB. Let the

arc with its chord be rotated round O through a very small $\angle a$ into the position A′DB′, the Center of Mass being now G′; then $\angle AMA' = AOA' = GOG' = a$. This rotation has been given for convenience of the construction: the result is the same as if the very small arc AA′ were transferred into the position BB′.

In the limit when AA′ is infinitely small GG′ is parallel to AB, the Center of Mass of AA′ travels from A to B along AB

$$\therefore \frac{GG'}{AB} = \frac{AA'}{\text{arc}},$$

$$i.\,e. \quad \frac{OG \times a}{AB} = \frac{r \times a}{\text{arc}}$$

$$\therefore OG = \frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

If $\angle AOG = \theta$ this becomes

$$OG = \frac{r \sin \theta}{\theta}. \tag{1}$$

For a semicircle this becomes

$$\frac{2r}{\pi}. \tag{2}$$

Cor. The Center of Mass of a sector, radius r and $\angle 2\theta$, being that of an arc radius $\frac{2}{3}r$, and $\angle 2\theta$ is given by:

$$\text{Distance from center} = \frac{2r \sin \theta}{3\theta}. \tag{3}$$

For a semicircular lamina this becomes $\frac{4r}{3\pi}$, (4)

θ being in this case $\frac{\pi}{2}$.

These last two might be investigated separately by the method of this article, by transferring an infinitely small triangle standing on an infinitely small arc of the circle and having its vertex at the center.

3. *To find the center of mass of a segment of a circle.*

Let ADB (Fig. 2), be the segment, and let the construction be as there made, except that G and G′ are now the Centers of Mass of the segments ADB and A′DB′. The portion transferred is now the infinitely small triangle AMA′ to BMB′. Since the Center of Mass of a triangle is $\frac{2}{3}$ of the median from the vertex, the distance gg' traveled by the Center of Mass of this triangle is, in the limit, $\frac{2}{3}$AB. The area of this triangle is $\frac{1}{2}$AM. A′M sin a, which in the limit becomes $\frac{1}{2}\left(\frac{AB}{2}\right)^2 \times a$

$$\therefore \frac{GG'}{\frac{2}{3}AB} = \frac{\frac{1}{2}\left(\frac{AB}{2}\right)^2 \times a}{\text{segment}}$$

$$\therefore \frac{OG \times a}{\frac{2}{3}AB} = \frac{AB^2 \times a}{8 \times \text{segment}}$$

$$\therefore OG = \frac{\text{cube of chord}}{12 \text{ times area of segment}}$$

$$\text{or } OG = \frac{2r \sin^2 \theta}{3(\theta - \sin \theta \cos \theta)}. \qquad (5)$$

2θ being the angle of the segment, and a segment being easily calculated as the difference between a sector and a triangle.

For a semicircular lamina this becomes $\frac{4r}{3\pi}$ as before.

4. The Center of Mass of a zone of a hollow sphere is found in an elementary manner by the elegant method of comparison with the right cylinder circumscribed to the sphere which seems due to Collignon. (See Collignon *Statique* p. 299; Loney *Statics and Dynamics*, etc.) The Center of Mass is on the axis of the zone and half way between the parallel planes. For a spherical cap, a particular case of zone, this result may easily be expressed in the form:— Distance of Center of Mass from center of sphere

$$= \frac{r \sin^3 \theta}{2(1 - \cos \theta)} \qquad (6)$$

r being the radius of the sphere and 2θ the angle of the cap.

For a hemispherical shell the distance from the center is $\frac{r}{2}$ (7)

From this last result (7) the Center of Mass of a lune of infinitely small angle may be immediately deduced by considering the hemispherical shell to be made up of such lunes. If x be the distance of the Center of Mass of one of the thin lunes from the center of the sphere, the Center of Mass of the hemisphere is reducible to that of the semicircular arc, radius x, formed by the Centers of Mass of the lunes. Its distance from the center of the sphere is $\therefore \frac{2x}{\pi}$ by (2); but we know this distance to be $\frac{r}{2}$ by (7)

$$\therefore \frac{2x}{\pi} = \frac{r}{2} \quad \therefore x = \frac{\pi r}{4}$$

or the distance of the Center of Mass of an infinitely thin lune from the center of the sphere is $\frac{\pi r}{4}$. (8)

From this we get the Center of Mass of a lune of any angle 2θ. Being the Center of Mass of an arc $\angle 2\theta$ and radius $\frac{\pi r}{4}$ its distance from the Center of the Sphere is by (1) $\frac{\pi r \sin \theta}{4\theta}$. (9)

It has been thought desirable to mention these results in passing, as being capable of easy determination without the use of the method at present under consideration. The case of the hemispherical shell must apparently be found independently of this method.

5. It will be interesting to investigate one of the foregoing results *e. g.*, *the Center of Mass of a lune of infinitely small angle* by the method of the present article or rather by a reversal of it, using as a necessary assumption the Center of Mass of a hemispherical shell.

Let ABC (Fig. 3) be a section of a hemispherical shell through its center, and perpendicular to its plane. Its Center of Mass G is

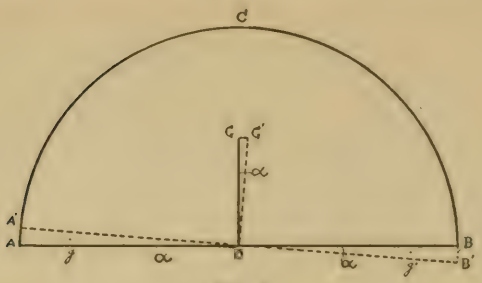


Fig. 3.

in this plane, and $OG=\frac{r}{2}$ by (7). If now the hemisphere be rotated round the diameter perpendicular to the plane of the paper, through a very small $\angle a$, so that the section becomes $A'CB'$ and the Center of Mass G', the result is the same as if a lune $\angle a$ were transferred over and attached to the opposite edge of the hemispherical shell, carrying its Center of Mass from g to g'.

We have then, $\frac{GG'}{gg'}=\frac{\text{lune}}{\text{hemisphere}}$ which becomes, when a is indefinitely diminished, $\frac{\frac{r}{2}\times a}{2Og}=\frac{a}{\pi}$ $\therefore Og=\frac{\pi r}{4}$ as above. (8)

The case of the spherical cap, deduced otherwise above (6), would afford a favorable example for the application of the method of this article, by transferring an infinitely thin lune from one semi-edge to the other.

The Center of Mass of a solid hemisphere is immediately reducible to that of a homogeneous hemispherical shell. Its distance from the center of the sphere is as is well known $\frac{3r}{8}$. (10)

6. Knowing this:—

To find the Center of Mass of a semicircular wedge of infinitely small angle.

Let the section ACB (Fig. 3) now represent a section of a solid hemisphere. OG now $=\frac{3}{8}r$ by (10) and AOA' is a section of a

solid wedge, transferred over to the position giving the section BOB′, and carrying its Center of Mass from g to g'; we have then

$$\frac{GG'}{gg'} = \frac{\text{wedge}}{\text{hemisphere}}$$

which becomes when a is indefinitely diminished $\frac{\frac{3}{8}r \times a}{2Og} = \frac{a}{\pi}$

$$\therefore Og = \frac{3\pi r}{16}. \qquad (11)$$

From this *the Center of Mass of a wedge of any angle 2θ, cut out of a solid sphere by the intersection of two diametral planes*, is easily found. Being the Center of Mass of an arc $\angle 2\theta$ and radius $\frac{3\pi r}{16}$ its distance from the center is by (1)

$$\frac{3\pi r \sin\theta}{16\theta}. \qquad (12)$$

The result in (11), like that in (8), might have been found by considering the hemisphere made up of an infinity of wedges of infinitely small angle. Also the Center of Mass of a wedge $\angle 2\theta$, result (12), might have been found by the method of this article.

7. *To find the Center of Mass of a segment of a solid sphere*, Let ACB (Fig. 4) be a section of the segment through its highest point, and through O the center of the sphere. G, the Center of Mass, is in this plane, and OG is perpendicular to AB.

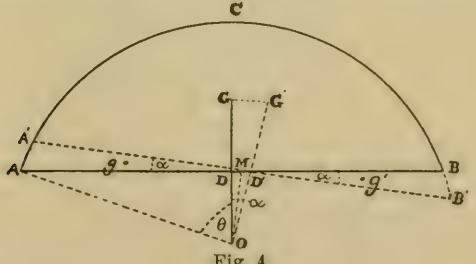


Fig. 4.

Let the segment be turned round an axis through O, perpendicular to the plane of the paper, through a small $\angle a$, bringing the Center of Mass to G′, and giving the section A′CB′, then $\angle AMA' = AOA' = GOG' = a$ and the result is the same as if the wedge whose section is AMA′ were transferred on to the other half of the face of the segment giving the section BMB′ and carrying its Center of Mass from g to g', then

$$\frac{GG'}{gg'} = \frac{\text{wedge}}{\text{segment}}.$$

Now the radius of the wedge is in the limit $r \sin\theta$, 2θ being the angle of the segment, $\therefore$ by (11) $gg' = 2\dfrac{3\pi(r \sin\theta)}{16}$; the volume of the wedge is $\dfrac{4}{3}\pi(r \sin\theta)^3 \times \dfrac{a}{2\pi} = \frac{2}{3}ar^3 \sin^3\theta$, and the volume of the segment is found by considering it as the difference between the sector and cone having their vertices at O, to be—

$$\frac{\pi r^3}{3}(1-\cos\theta)^2(2+\cos\theta)$$

$$\therefore \frac{GG'}{gg'} = \frac{\text{wedge}}{\text{segment}}, \text{ becomes in the limit—}$$

$$\frac{OG \times a}{\frac{3}{8}\pi r \sin\theta} = \frac{\frac{2}{3}ar^3 \sin^3\theta}{\frac{\pi r^3}{3}(1-\cos\theta)^2(2+\cos\theta)}$$

$$\therefore OG = \frac{3r \sin^4\theta}{4(1-\cos\theta)^2(2+\cos\theta)}$$

$$= \frac{3r(1+\cos\theta)^2}{2+\cos\theta}. \qquad (13)$$

If h is the height of the segment this result may be written in the form $OG = \dfrac{3(2a-h)^2}{4(3a-h)}$. (14)

The geometrical investigation of limiting values has been avoided, it is hoped successfully, by using a rotation as equivalent to actual transfer of the infinitesimal portion. In the present example, however, it seems advisable to go somewhat into detail.

D (Fig. 4) being the intersection of AB and OG, AD=DB; the $\angle$ DOM$=\frac{\alpha}{2}$ being infinitesimal, DM is infinitesimal, $\therefore$ AM =MB ; similarly A′M=MB′ ; $\therefore$ the centers of the circular sections AB and A′B′ are at M, $\therefore$ all circles on the surfaces of the wedges MAA′, MBB′, passing between A and A′, B and B′, and through the vertices of the wedges, have in the limit the common center M : their radii, though not absolutely equal, differ by infinitesimals from $r \sin \theta$ $\therefore$ in the limit, the volume and Center of Mass of each wedge may be taken as if it were cut by diametral planes including an infinitely small angle, out of a sphere of radius $r \sin \theta$.

8. *If a circular lamina of uniform mass, be gradually pushed over another of the same diameter and thickness, to find the Center of Mass of the portion of either uncovered by the other, when they are just about to coincide.*

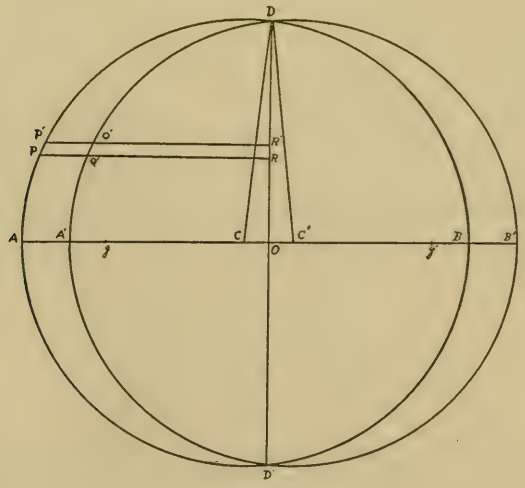


Fig. 5

Let ADBD′ and A′DB′D′ (Fig. 5) be two equal circles. The question is plainly the same as to find the Center of Mass of the area D AD′A′ (or DB′D′B), when CC′ (=AA′=BB′) the distance between the centers of the circles is infinitesimal.

Let g be the Center of Mass of DAD′A′ and the corresponding point g' that of DB′D′B.

The distance PQ intercepted between the two circles on any parallel to AB is evidently constant and =CC′, because in a motion of pure translation all points of a body move through equal distances, ∴ the area of any thin strip PQQ′P′ intercepted between two such parallels =CC′×RR′ ∴ in the limit the area of DAD′A′ =CC′× the limit of DD′,=CC′×$2r$.

If now DAD′A′ be transferred to the position DB′D′B, carrying its Center of Mass from g to g', the Center of Mass of the whole circle DAD′B moves from C to C′ as the two circles now plainly coincide, ∴ in the limit,

$$\frac{gg'}{CC'}=\frac{\pi r^2}{2r.CC'},\ \therefore gg'=\frac{\pi r}{2}.$$

But in the limit gg' is twice the distance of the required Center of Mass from the center of the circle, ∴ this distance is $\frac{\pi r}{4}$. (15)

Note the correspondence between the results in this case and the case of a lune of infinitely small angle.

If the circles overlap in any position we can similarly find the Center of Mass of the uncovered portion of either.

If the angle CDC′ be 2θ and O the middle point of CC′, the area of DAD′A′ is $r^2(2\theta+\sin 2\theta)$

$$\therefore \frac{2Og}{2r\sin\theta}=\frac{\pi r^2}{r^2(2\theta+\sin 2\theta)}$$

$$\therefore Og=\frac{\pi r\sin\theta}{2\theta+\sin 2\theta}. \quad (16)$$

On proceeding to the limit ($\theta=0$), this gives the same result as above (15).

9. *If two equal spheres intersect to find the Center of Mass of homogeneous matter filling either of the enclosed spaces not common to both, when the spheres are about to coincide.*

Let the *whole* enclosed space be supposed filled with homogeneous matter and let Fig. 5 represent a section of it through the centers C and C′ of the spheres, g and g' now representing the required Centers of Mass, namely of the portions of which DAD′A′ and DB′D′B are sections. The volume of DAD′A′ in the limiting position, is by reasoning similar to that used in the case of the overlapping laminæ, $= CC' \times$ area of the circle of intersection of the spheres in the limiting position, $= CC' \times \pi r^2$. If now DAD′A be supposed transferred to the position, of DB′D′B, carrying its Center of Mass from g to g', the Center of Mass of the sphere DAD′B travels from C to C′. We have then—

$$\frac{gg'}{CC'} = \frac{\frac{4}{3}\pi r^3}{CC' \times \pi r^2}$$

$$\therefore gg' = \tfrac{4}{3}r$$

$\therefore$ the distance of the required Center of Mass from the center of the sphere is $\dfrac{2r}{3}$. (17)

If the spheres intersect in any position, we can, as in the case of the circular laminæ just preceding, find the Center of Mass of the part of either not common to the other.

Let Fig. 5 represent a section of the spheres through their centers, g and g' being now Centers of Mass of the solids DAD′A′, DB′D′B, and $\angle CDC' = 2\theta$.

The volume of DAD′A′ $= \dfrac{2\pi r^3 \sin\theta}{3}(2+\cos^2\theta)$[1]

$$\therefore \frac{2Og}{2r\sin\theta} = \frac{\frac{4}{3}\pi r^3}{\frac{2\pi r^3 \sin\theta(2+\cos^2\theta)}{3}}$$

[1] Vol. DAD′A′ = Sphere DAD′B − 2 Segment DA′D′ = Sphere − 2 Sector C′DA′D′ + 2 Cone C′DD′.

$$\therefore Oy = \frac{2r}{2+\cos^2\theta}. \qquad (18)$$

On proceeding to the limit ($\theta=0$) this gives same result as above (17).

10. *If an ellipsoid, (axes a, b, c, in order of magnitude), receive a displacement of pure translation along its a axis, to find the Center of Mass of homogeneous matter filling the space vacated, (or the new space occupied), when the displacement is infinitesimal.*

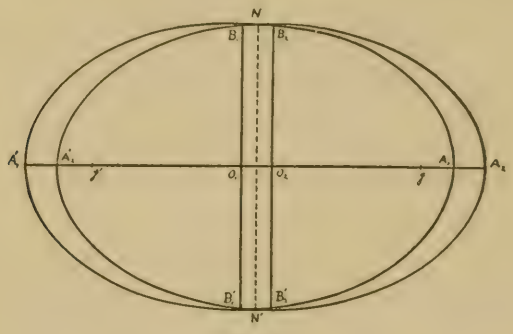


Fig. 6.

Let $A_1B_1A_1'B_1'$, $A_2B_2A_2'B_2'$ (Fig. 6) be sections through the a and b axes of the ellipsoids in the two very near positions, the ellipsoids intersecting in the plane of which NN' is a section. Suppose the whole volume $A_2NA_1'N'$ occupied by homogeneous matter, the volume of either ellipsoid is $\frac{4}{3}\pi abc$, and as the linear displacement of all points on the ellipsoid is the same and $=A_1A_2=O_1O_2=A_1'A_2'$, it is easily seen that the volume of $NA_2N'A_1$ or $NA_1'N'A_2'$ is in the limit the product of the area of the plane of intersection by the linear displacement $=\pi bc\times O_1O_2$.

Consider the ellipsoid A_1A_1'. Its Center of Mass is O_1 and if the portion $NA_1'N'A_2'$ be supposed transferred into the position $NA_2N'A_1$, carrying with it its Center of Mass from y' to y the Center of Mass of the ellipsoid evidently becomes O_2

$$\therefore \frac{gg'}{O_1O_2} = \frac{\frac{4}{3}\pi abc}{\pi bc \times O_1O_2}$$

$$\therefore gg' = \tfrac{4}{3}a.$$

But gg' in the limit is twice the distance O_1g' of the required Center of Mass from the center of the ellipse,

$$\therefore \text{this distance is } \tfrac{2}{3}a. \qquad (19)$$

This result gives the position of the pole of an ellipsoid of magnetisable matter, in a uniform field in the direction of the a axis of the ellipsoid.[1]

[1] See Maxwell's Treatise on Electricity, Vol. II, Section 437.

PRELIMINARY LIST OF BIRDS OF BOULDER COUNTY, COLORADO

By Junius Henderson

This list of 160 species must be considered strictly preliminary and very incomplete. The latest available list for the state—that of Leander Keyser in "Birds of the Rockies"—records 389 species for the state. Many of these undoubtedly visit Boulder County which have not come to the attention of the writer. Species reported upon doubtful identification have either been omitted or followed by an interrogation point. The plan has been to first list the species of which specimens taken in the county are now in the University collection, adding to these the species not in the collection but mentioned in the manuscript lists of the writer and of Walter Blanchard, those reported by C. W. Rowland and James Cowie, those reported by Dennis Gale in Bendire's "Life Histories of North American Birds," H. D. Minot's list published in a bulletin of the Nuttall Ornithological Club many years ago, and those reported by Mrs. Maxwell, C. J. Hersey, H. G. Smith, R. D. Campbell, Wm. A. Sprague, R. C. McGregor, A. W. Anthony and Prof. W. W. Cooke, in the three Bulletins prepared by the latter for the State Agricultural College, on the "Birds of Colorado." The name in parentheses following the name of a species indicates the person on whose authority it is reported, and all species not so followed by a name are now represented in the collection. Most of the species in the collection were mounted by L. C. Bragg, of Boulder, without expense to the institution except for materials required, and many of them were taken by him. A number of species now in his hands may be in the collection before the publication of this paper.

Gavia imber—Common loon.
Gavia arctica—Black-throated loon (Rowland).
Stercorarius pomarinus—Pomarine Jaeger (Mrs. Maxwell).
Larus delawarensis—Ring-billed gull.
Hydrochelidon nigra surinamensis—Black tern (Rowland).
Rissa tridactyla—Kittiwake (Mrs. Maxwell).
Xema sabini—Sabine gull (Cooke).
Merganser serrator—Red-breasted merganser (Rowland).
Merganser lophodytes cucullatus—Hooded merganser (Rowland).
Anas boschas—Mallard duck (Henderson).
Chaulelasmus streperus—Gadwall duck (Rowland).
Mareca americana—Baldpate duck (Rowland).
Nettion carolinensis—Green-winged teal (Henderson).
Querquedula discors—Blue-winged teal (Henderson).
Querquedula cyanoptera—Cinnamon teal (Rowland).
Spatula clypeata—Shoveller (Rowland).
Dafila acuta—Pintail or sprigtail duck (Rowland).
Aythya americana—Redhead duck (Rowland).
Aythya vallisneria—Canvas-back duck.
Aythya sp?—Greater or lesser scaup duck (Rowland).
Aythya collaris—Ring-necked duck (Rowland).
Clangula sp?—Goldeneye (Rowland).
Aix sponsa—Wood duck (Cowie).
Charitonetta albeola—Bufflehead.
Harelda hyemalis—Old squaw duck (Rowland).
Erismatura jamaicensis—Ruddy duck (Rowland).
Chen hyperborea—Lesser snow goose (Rowland).
Chen hyperborea nivalis—Greater snow goose (Rowland).
Branta canadensis—Canada goose (Rowland).
Ardea herodias—Great blue heron.
Ardea candidissima—Snowy egret (Hersey).
Nycticorax nycticorax naevius—Black-crowned night heron (Bragg).
Rallus virginianus—Virginia rail (Rowland).
Fulica americana—American coot, mudhen (Henderson).
Recurvirostra americana—American avocet (Cowie).
Himantopus mexicanus—Black-necked stilt (Rowland).
Aegialitis vocifera—Killdeer.
Philohela minor—American woodcock (Smith, Rowland).
Gallinago delicata—Wilson snipe (Rowland).
Micropalama himantopus—Stilt sandpiper (Rowland).
Tringa fuscicollis—White-rumped sandpiper (Rowland).
Totanus flavipes—Yellow-legs (Rowland).
Totanus melanoleucus—Greater yellow-legs (Rowland).
Actitis macularia—Spotted sandpiper (Blanchard).

Numenius longirostris—Long-billed curlew (Rowland).
Lagopus leucurus—White-tailed ptarmigan.
Dendragapus obscurus—Dusky or blue grouse (Henderson).
Colinus virginianus—Bob-white, quail (Henderson).
Zenaidura macroura—Mourning dove, turtle dove (Henderson).
Cathartes aura—Turkey vulture (Rowland).
Circus hudsonius—Marsh hawk (Rowland).
Accipiter velox—Sharp-shinned hawk (Rowland).
Accipiter cooperii—Cooper hawk.
Aquila chrysaëtos—Golden eagle.
Haliæetus leucocephalus—Bald eagle (Blanchard).
Falco mexicanus—Prairie falcon (Blanchard).
Falco peregrinus anatum—Duck hawk (Rowland).
Falco sparverius—American sparrow hawk.
Buteo calurus—Western red-tailed hawk.
Strix pratincola—American barn owl (Rowland).
Asio wilsonianus—Long-eared owl (Rowland).
Asio accipitrinus—Short-eared owl (Rowland).
Nyctala acadica—Saw-whet owl (Gale).
Megascops flammeola—Flammulated owl (Sprague specimen, very rare).
Megascops asio maxwelliae—Rocky Mountain screech owl (Gale).
Speotyto cunicularia hypogaea—Burrowing owl (Henderson).
Bubo virginianus pallescens—Western horned owl.
Nyctea nyctea—Snowy owl (Rowland).
Glaucidium gnoma—Pygmy owl (Bragg).
Ceryle alcyon—Belted kingfisher (Henderson).
Colaptes cafer collaris—Red-shafted flicker (Henderson).
Sphyrapicus varius nuchalis—Red-naped sapsucker (Rowland).
Melanerpes erythrocephalus—Red-headed woodpecker (Henderson).
Melanerpes torquatus—Lewis woodpecker.
Dryobates villosus monticola—Rocky Mountain hairy woodpecker (Anthony)
—type locality.
Phalænoptilus nuttallii—Poorwill (Blanchard).
Chordeiles virginianus henryi—Western night-hawk (Henderson).
Aeronautes melanoleucus—White-throated swift (Blanchard).
Selasphorus platycercus—Broad-tailed hummingbird (Henderson).
Selasphorus rufus—Rufous hummingbird (Gale).
Tyrannus tyrannus—Kingbird.
Sayornis saya—Say's phoebe.
Contopus borealis—Olive-sided flycatcher.
Contopus richardsonii—Western wood pewee.
Empidonax traillii—Traill flycatcher (Minot).
Otocoris alpestris leucolaema—Pallid or desert horned lark (Henderson).
Pica pica hudsonica—Black-billed magpie.

Cyanocitta stelleri diademata—Long-crested jay (Henderson).
Perisoreus canadensis capitalis—Rocky Mountain jay (Henderson).
Perisoreus canadensis—Canada jay (Blanchard and others)?
Molothrus ater—Cowbird (Blanchard).
Nucifraga columbiana—Campbird, Clarke crow.
Corvus cryptoleucus—White-necked raven (Campbell).
Xanthocephalus xanthocephalus—Yellow-headed blackbird.
Scoleocophagus cyanocephalus—Brewer blackbird.
Quiscalus quiscula æneus—Bronzed grackle (Blanchard).
Icterus galbula—Baltimore oriole (Rowland)?
Icterus bullocki—Bullock oriole (Henderson).
Sturnella magna neglecta—Western meadowlark (Henderson).
Coccothraustes vespertinus montanus—Western evening grosbeak.
Pinicola enucleator montana—Rocky Mountain pine grosbeak (Henderson).
Carpodacus mexicanus frontalis—House finch.
Loxia curvirostra stricklandi—Mexican crossbill (Minot). Possibly just over Gilpin County line.
Leucosticte australis—Brown-capped leucosticte, rosy finch (Henderson).
Leucosticte tephrocotis—Gray-crowned leucosticte (Sprague).
Acanthis linaria—Redpoll (Henderson).
Acanthis linaria rostrata—Greater redpoll (Sprague)?
Astragalinus tristis—American goldfinch.
Passer domesticus—English sparrow.
Passerina nivalis—Snowflake (Cooke).
Zonotrichia leucophrys—White-crowned sparrow (Minot).
Spizella monticola ochracea—Western tree sparrow (Henderson).
Spizella socialis—Chipping sparrow (Minot). Possibly western form.
Junco caniceps—Gray-headed junco (Henderson).
Junco hyemalis connectens—Intermediate junco (McGregor).
Junco mearnsi—Pink-sided junco (Henderson).
Junco aikeni—White-winged junco (McGregor).
Junco annectens—Ridgeway junco (McGregor).
Melospiza lincolnii—Lincoln sparrow.
Pipilo maculatus megalonyx—Spurred towhee.
Oreospiza chlorura—Green-tailed towhee (Blanchard, Minot).
Cyanospiza amoena—Lazuli bunting, painted finch (Henderson).
Calamospiza melanocorys—Lark bunting, white-winged blackbird, often mistaken for bobolink (Henderson) .
Calcarius ornatus—Chestnut-collared longspur (Cooke).
Piranga ludoviciana—Louisiana tanager.
Petrochelidon lunifrons—Cliff swallow (Blanchard).
Hirundo erythrogastra—Barn swallow.
Ampelis garrulus—Bohemian waxwing (Bragg).
Lanius borealis—Northern shrike, butcher-bird (Henderson).

Vireo gilvus—Warbling vireo (Minot).
Vireo solitarius plumbeus—Plumbeous vireo (Minot).
Dendroica æstiva—Yellow warbler, yellow summer-bird.
Dendroica auduboni—Audubon warbler (Minot).
Helminthophila virginiæ—Virginia warbler (Minot).
Helminthophila celata—Orange-crowned warbler (Minot).
Helminthophila peregrina—Tennessee warbler (Minot).
Mniotilta varia—Black and white warbler, very rare (Minot).
Icteria virens longicauda—Long-tailed chat (Henderson).
Setophaga ruticilla—Redstart (Keyser).
Seiurus aurocapillus—Overbird (Minot).
Seiurus noveboracēnsis notabilis—Grinnell water-thrush (Cooke).
Anthus pensilvanicus—Pipit (Minot).
Geothylpis tolmiei—Macgillivray warbler (Minot).
Cinclus mexicanus—Water ouzel, dipper.
Galeoscoptes carolinensis—Catbird.
Salpinctes obsoletus—Rock wren.
Troglodytes aëdon aztecus—Aztec or house wren.
Catherpes mexicanus conspersus—Cañon wren (Anthony).
Sitta carolinensis—White-breasted nuthatch (Blanchard).
Sitta pygmæa—Pygmy nuthatch (Blanchard).
Parus articapillus septentrionalis—Long tailed chickadee (Blanchard).
Parus gambeli—Mountain chickadee (Henderson).
Myadestes townsendii—Townsend Solitaire (Blanchard).
Hylocichla ustulata swainsonii—Olive-backed thrush.
Hylocichla aonalaschkae—Dwarf thrush (Sprague)?
Hylocichla guttata auduboni—Audubon hermit thrush (Minot).
Merula migratoria propinqua—Western robin (Henderson).
Saxicola œnanthe leucorhoë—Wheatear (Minot)?
Sialia arctica—Mountain bluebird.
Sialia sialis—Bluebird (Gale).

THE COTYLEDONS AND LEAVES OF CERTAIN PAPILIONACEAE

By Francis Ramaley

Nature of the Cotyledon. The view that cotyledons are to be considered as leaves has long been held and botanists and others are accustomed to speak of cotyledons as "seed leaves." Such a conception is natural enough from a mere examination of external appearances. Doubtless Goethe's doctrine of metamorphosis, which has dominated much of the botanical teaching of the past, is also partly responsible for this view. But although the cotyledons have been referred to as "seed leaves" the differences which they show have been frequently noted. A recent suggestion by Lyon[1] is that the cotyledon was originally a haustorial organ which became modified for purposes of storage and, in some cases, becoming epigean through the elongation of the hypocotyl, took on the appearance and function of a foliage leaf. According to this view similarity of cotyledons and foliage leaves has been brought about through adaptation to similar conditions of life but the two structures are essentially different. The haustorial nature of the cotyledon has been referred to recently by Thistleton-Dyer,[2] and others, and has doubtless been recognized by many botanists. Thistleton-Dyer's recent suggestion of the value of anatomical study of cotyledons has led to the publication, at this time, of the results of some studies which have been carried on by the writer during the past two years.

Material. Seedlings were grown of a considerable number of Colorado Papilionaceae. These were taken at different periods of development and preserved in alcohol. A large amount of similar

(1) Lyon, in Postelsia, 57-86. 1902.
(2) Thistleton-Dyer, in Ann. of Bot., **16**: 553. 1902.

alcoholic material of plants belonging to other families is on hand at the present time and a report on a study of it will be ready in a few months.

Species examined. In the species selected for study the cotyledons are always epigean. They are not greatly thickened for storage purposes and they remain active for some time after appearing above ground. They all increase in size more or less and assume the appearance of foliage leaves, but they are different in shape from the true foliage leaves. Although sessile in most species, in some there are distinct stalks which become elongated with increase in age. The following species were studied: *Astragalus adsurgens* Pall., *A. carolinianus* Linn., *A. crassicarpus* Nutt., *A. flexuosus* (Hook.) Dougl., *A. hypoglottis* Linn., *A. racemosus* Pursh, *Aragallus deflexus* (D. C.) Heller, *Aragallus spicatus* (Hook.) Rydb., *Trifolium dasyphyllum* Torr. and Gray, *Robinia pseudacacia* Linn., *Amorpha canescens* Pursh, *Lupinus pusillus* Pursh, *Glycyrrhiza lepidota* Pursh, *Psoralea hypogaea* Nutt., *Hedysarum mackenzii* Richards, and *Petalostemon candidus* Michx.

External Morphology of Cotyledons and Leaves. It has been frequently shown[1,2,3] that no general relation can be established between the shapes of cotyledons and foliage leaves. It will not, therefore, be useful to give an extended account of the differences noted. The cotyledons of the plants studied are ovate or oblong, generally sessile or nearly so, and in most cases somewhat asymmetrical and slightly thicker than the foliage leaves. They increase in size after emerging from the seed coat, usually growing to twice or three times their former length and breadth. The foliage leaves are always eventually compound but usually the first two or more leaves are simple and frequently the succeeding ones have fewer leaflets than the characteristic leaves produced by the adult plant. *Lupinus pusillus* forms an exception to the rule as its first leaves are palmate.

(1) Lubbock. On Seedlings. 1: 9. 1892.
(2) Klebs. Beitrage zur Morph. und Biol. der Keimung. Pfeffer's Untersuchungen aus dem Bot. Inst. zu Tubingen. 1: 536. 1885.
(3) Ramaley. Seedlings of Certain Woody Plants. Minn. Bot. Stud. 2: 84. 1899.

Stalks of Cotyledons and Leaves. The cotyledons are sessile in most of the plants studied but they are stalked in *Psoralea hypogaea*, *Glycyrrhiza lepidota*, *Trifolium dasyphyllum* and *Petalostemon candidus*. In these four cases a comparative study was made of the anatomy of the stalks of the cotyledons and foliage leaves. Great differences were observed in all cases. The stalk of the cotyledon is not so cylindrical as that of the leaf (Figs. 1 and 3). The vascular tissue is placed near the center of the stalk, in a single bundle, or in two which are close together. In the leaf-stalk there are three or five bundles (Figs. 2 and 4), arranged in a partial circle. The writer has previously[1] noted just this same difference between the stalks of cotyledons and leaves in *Delphinium*.

Epidermal Cells. The epidermal cells, seen in surface view, sometimes have a very wavy outline (Fig. 6). This is especially the case with the foliage leaves. In the cotyledons cells with this wavy appearance occur in only a few species. More usually the walls are nearly straight (Fig. 5). The following plants conform to the general rule just stated: *Astragalus crassicarpus*, *Astragalus racemosus*, *Robinia pseudacacia*, *Amorpha canescens*, *Aragallus deflexus*, *Arayallus spicatus*, *Petalostemon candidus*, *Hedysarum mackenzii*, and *Glycyrrhiza lepidota*. In the following plants there is sometimes a tendency for the respective parts to be like those named above, but more usually the epidermis of both the leaves and cotyledons is wavy: *Astragalus adsurgens*, *A. carolinianus*, *A. flexuosus*, *A. hypoglottis*, *Psoralea hypogaea*. *Trifolium dasyphyllum* and *Lupinus pusillus* have the epidermis similar in the leaf and cotyledon but in this case the cells are not wavy.

Stomata. There are no stomata on the upper surface of the leaf in *Robinia pseudacacia* and *Amorpha canescens*, though they are present on both surfaces of the cotyledon. All other plants examined have stomata on both surfaces of leaves aud cotyledons.

Trichomes. No trichomes were found on any of the cotyledons studied. It must not be supposed from this statement that they are never found on cotyledons for they are present on the cotyledons of

(1) Ramaley. The Seed and Seedling of the Western Larkspur. Minn. Bot. Stud. 2: 417. 1900.

many plants. The trichomes of the leaves, in the plants examined, are frequently abundant on the lower surface, but few or none are present on the upper surface. Only *Petalostemon candidus* and *Glycyrrhiza lepidota* are without trichomes.

Internal Structure of Leaf and Cotyledon. The cotyledon (Fig. 9) is generally thicker than the foliage leaf. The palisade is composed of rather broad, cylindrical cells usually forming about three rows. The spongy tissue consists of cells which are spherical or nearly so. No very large air spaces are present. Vascular tissue is poorly developed and there is usually no prominent mid-rib. In the foliage leaf (Fig. 8) the palisade cells are narrower and form about two rows. Air spaces of considerable size are found in the spongy tissue which is made up of cells quite irregular in form. A prominent mid-rib is present and the veins are generally larger than those of the cotyledon. While the structures just described are the rule certain exceptions must be noted. Thus the spongy tissue of the leaf of *Amorpha canescens* resembles that described above as typical for cotyledons and the spongy tissue of the cotyledons in *Robinia pseudacacia* shows a similarity to the structure usually seen in foliage leaves. Cells without chlorophyll occur in the palisade of of the leaves of *Psoralea hypogaea* but not in the cotyledons of the same species. Large cells with brownish contents are found in the leaves and leaf-stalks of *Hedysarum mackenzii* and *Glycyrrhiza lepidota* (Fig. 7), but are entirely absent from the cotyledons. These cells have been examined and described by Bokorny[1] for *Hedysarum mackenzii*. In *Glycyrrhiza* the secretory cells are of the same kind but occur only in the palisade region, not scattered. In *Petalostemon candidus* there are numerous, spherical, multicellular glands in the mesophyll and particularly at the margins of the leaf. None are present in the cotyledons.

Summary and Conclusions. It has long been held that there is no general relation in shape between the cotyledons and leaves in the same species of plant. This view is confirmed by the studies here recorded. In the plants examined, all of which have cotyledons which

(1) Fide Solereder. Compar. Anat. der Dicotyledonen, 296. 1899.

function for a time as leaves, the anatomical structure is strikingly different in cotyledons and leaves. Every plant examined has glabrous cotyledons, while leaves of the same species nearly always bear trichomes. The cotyledons, in all cases, have stomata on both surfaces, but some of the leaves have stomata only on the lower surface. Frequently there is a difference in the shape of the epidermal cells. Where this occurs the cell outlines are more wavy in the leaves than in the cotyledons. The reverse condition never occurs. The spongy tissue is frequently much more loose in the leaves than in the cotyledons. Glandular cells, or cells free from chlorophyll, may occur in the leaves but never in the cotyledons. The stalk of the cotyledon, when present, has a structure different from that of the leaf-stalk. The writer does not wish to generalize from the facts given here, but intends to continue the study in other plant families.

Explanation of Figures

1. *Psoralea hypogæa;* diagram of cross section of stalk of cotyledon. The two vascular bundles are close together. Xylem dotted, stereom black, phloem white.

2. *Psoralea hypogæa;* diagram of cross section of leaf-stalk. The vascular bundles, five in number, are arranged in a circle with the phloem facing outward.

3. *Trifolium dasyphyllum;* diagram of cross section of the stalk of the cotyledon.

4. *Trifolium dasyphyllum;* diagram of cross section of leaf-stalk.

5. *Hedysarum mackenzii;* epidermis of lower surface of cotyledon×160.

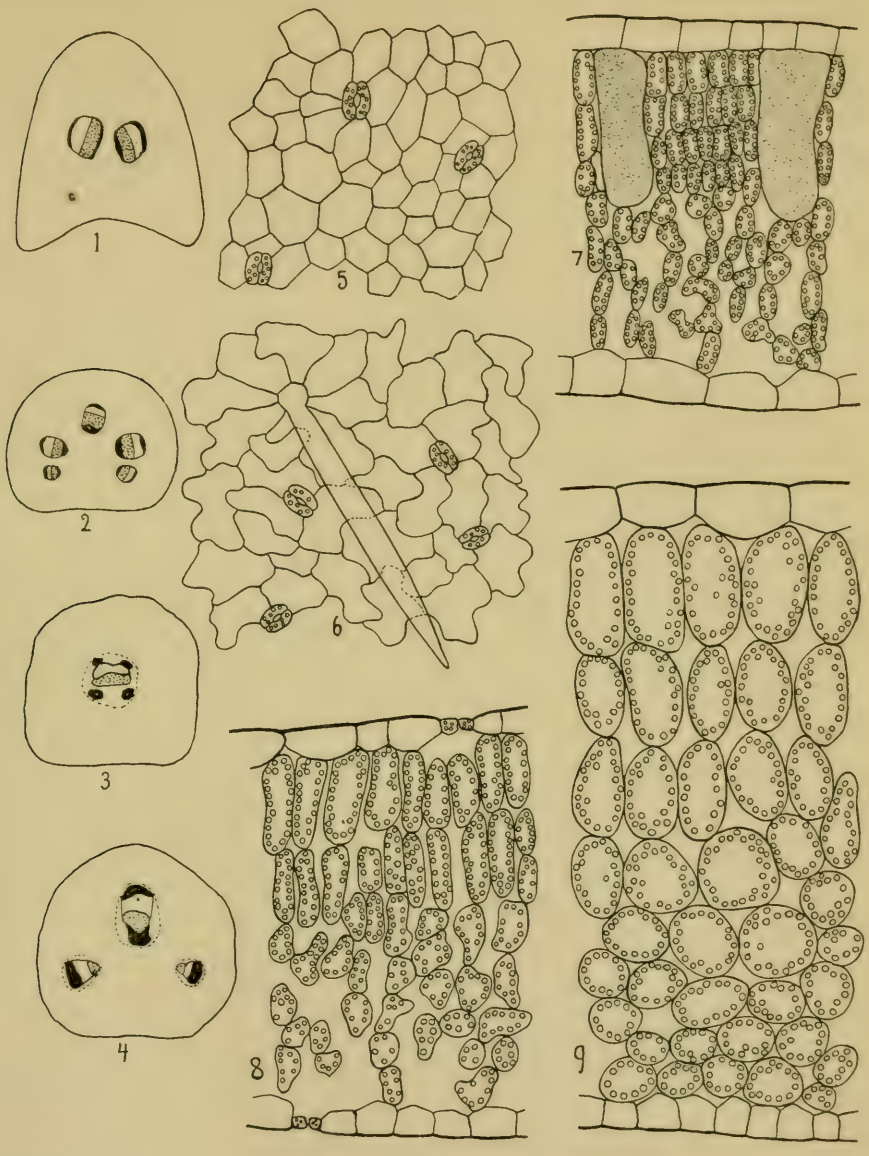
6. *Heydsarum mackenzii;* epidermis of lower surface of foliage leaf×160. The wavy outline of the cells gives them an appearance very different from the epidermal cells of the cotyledon. (Fig. 5.)

7. *Glycyrrhiza lepidota;* vertical section of leaf×160. The large cells in the palisade region are filled with brownish granular contents. Such cells do not occur in the cotyledons of the same plant.

8. *Astragalus flexuosus;* vertical section of leaf×160. The palisade is composed of narrow, cylindrical cells and the spongy tissue has many cavities.

9. *Astragalus flexuosus;* vertical section of cotyledon×160. The palisade is composed of about three layers of cells which are much broader than those in the foliage leaf. The air cavities of the spongy tissue are not large.

COTYLEDONS AND LEAVES OF CERTAIN PAPILIONACEAE



THE BASIS OF SOCIALITY[1]

By Arthur Allin

To join the hue and cry against Spencer's analogical comparison of society with an organism, though popular in certain sociological circles, is paying but scant respect to the real value of one of the pioneer attempts to secure a scientific basis for that foundling science—sociology. As a *heuristisches Princip* its most perdurable value lies in the fact that it has materially aided in the consideration of the sociological as a continuation of the biological. Strictly speaking, the biological probably includes human interaction or the social phenomena of human life, but for the purposes of a division of labor in the scientific world there has been a strong unconscious, though some unkind critic will say all too conscious, current in favor of founding a new discipline and department of human knowledge. Certainly the problems are ample enough to justify the division, and despite the similarity of laws the differences are sufficient to mark the boundaries of a new province of scientific research. Darwin's *Origin of Species* was really a description of organic technology, and the extra-organic sense and motor organs of social evolution are but the extensions of the tools and instruments which were so successful in the organic conflict. Organic heredity is continued in social heredity, the instinctive giving way, as second in importance, to oral and written tradition and the transmission of institutional life. The organic gains of the individual become objectified and perpetuated for all time in the environment, and an attainable object of possession for all socially minded people. The language of gestures of organic biology becomes the language of symbols with its priceless economy of time and labor. These laws and many others provide

(1) Reprinted by courtesy of the Editor of the American Journal of Sociology. Vol. VIII, July, 1902.

ample scope for the most untiring laborer and the most brilliant genius in the field of research. Spencer's analogy, therefore, is insufficient, and the attempt to base sociology on the specifications laid down for an organism is but little superior to the attempts of certain other sociologists who find all sociology bound up in the consciousness of kind or in the psychological process of imitation. Spencer, however, did point to the continuity of law as is evidenced in the biological and sociological worlds. Instead of an organism he could have used a species with much greater effect, for in a species are found, although in a crude and rudimentary stage, the first beginnings of social life.

One of the most striking, and yet at the same time one of the least observed, facts about specific action is the pre-eminence of the specific as such. The individual is secondary to the species. Instincts, which are characteristically the grand trunk line of transmission and continuity in the lower orders of the zoölogical series, are peculiar and very important in this, that they are always in their origin and bloom for the benefit of the *species* to which the animal may belong which possesses the instinct. They are of benefit to the individual only secondarily, in so far as that individual may be of benefit to the species. The mother gives up her life for the child. She dies, but the child, and through it the species, lives. The salmon struggles up the Columbia river for a thousand miles, is torn and battered by the rocks and waterfalls on the long and weary journey, lays its eggs, and dies; but the race lives on, although at the loss and sacrifice of one of its best members. The long history of the mammalia or *mothers* is a record of innumerable such examples. Of course, it is not necessarily true that the individual performs an instinctive act *with the consciousness that* the species may be benefited, but the persistent fact remains that in the long run only those species and individuals survive which act in such a way that the species may be further propagated. Instincts are always for species or race preservation. They are specific, altruistic, other-regarding, profoundly social. They may not be all consciously such, but in their origin and bloom they are in their final import intensely social.

It is a question of survival. It is a question of propagation and of the safety and welfare of the propagated. The individuals of a species which do not propagate obviously nullify the probability of like descendants. That which militates against the species thereby militates against the survival of the members of that species. The species that survives is characterized by the fact that its members act in such a manner that descendants are provided, and also provided *for* in some way or other. The goal of their activities is the young and their welfare. The young are heirs of all efforts directly or indirectly (*Erziehung, eine Fortsetzung der Erzeugung*). In the highest mammalian species, man, art, religion, and science are, in the long run, directly or indirectly, means for more certain perpetuation of the species and the more certain welfare of the same. The rank of a species is determined by the degree of such care for the young. The survival of the fittest means the survival of the parental, and all efforts are to be judged according to a parental standard. The greatest good to the greatest number must also be interpreted in a similar manner, not as the greatest happiness of the greatest number, but as such parental conduct, direct or indirect, as will be most conducive to the propagation and welfare of the species. As Herbert Spencer says, the continued life of the species is in every case the end to which all other ends are secondary (*Principles of Sociology*, Vol. I, p. 591). Through many stages of provincial patriotism and group-exclusiveness we have forged on until on the not far distant sky-line we see a state outlined where all humanity is our fatherland. All conduct is judged by nature according to the standard of survival.

In an organism, to recur to the Spencerian analogy, the conduct of the parts is determined by the welfare of the whole. That part which is detrimental to the whole organism is suicidal in tendency either immediately or mediately through the destruction of the whole organism. The safety of the parts lies in their general social efficiency. Their existence and perpetuation lie in their service to the general organization of which they form a part. To this extent an organism is similar to society, and to this extent is Spencer's analogy

pertinent. Neither the science of sociology nor the science of ethical conduct, it is evident, can be drawn from the individual as such. Plato, it seems, saw this when he endeavored to derive the true significance of justice and righteousness from the state, and not from the individual.

It seems plain, then, that the individual as such has no rights. The rights he may possess are attained by him through social service. It is through society that he acquires whatever rights he may claim. There was more sociological truth than cynicism in the reply of the French judge to a prisoner who excused his crime on the plea that "a man must live." "Pardon me," came the rejoinder, "but I don't see the necessity." The inalienable rights of the individual are *nil* excepting in so far as society may grant them. The individual pure and simple, *der Mensch überhaupt*, is a fiction. All which tends to survive is an organized whole of interacting parts.

The basis of sociality and the material of the science of sociology are therefore found in the interaction of parts which constitute a more or less organized whole. The organized whole, or society, is not something different from the interacting parts; the interacting parts *are* the society. The social is not the product of the interaction; it *is* the interaction. Each part is a partner or *socius* or *Theilnehmer*, the service or sociality of one part being complementary to the service of the other parts. Thus the social is reciprocal service. The social arises when the *Nebeneinander* becomes the *Miteinander*, when the anatomical becomes the physiological. The sociality consists in the correlated, co-ordinated activity of the integrated parts. Sociality is conduct, service rendered, not a consciousness of kind nor a feeling of sympathy, excepting in so far as they may be useful for the conduct of the parts.

Pitting the individual against society is an instance of crude sociological thought. Its ambiguity is at once manifest when one remembers that society does not exist as something separate from the integrated functions of the parts. It may be said that in the long run only those parts are allowed to exist which contribute to the social or organic welfare. The case in which possibly an individual

may be pitted against society is when the function of a part is prejudicial to organic survival. Such conduct is manifestly suicidal and, comparatively speaking, non-transmissible. It is, however, still a matter of sociality in that it is the service rendered by a part in an organized whole. It is, however, to be classed in what may be termed pathological as opposed to normal sociology. The truest part of man, the best and most righteous, is that which is most specific and most altruistic, that which contributes most to social organic welfare, which again must be defined in terms of survival of well-provided-for progeny. True selfishness or sin is that service rendered the whole which is for the individual's own immediate benefit and which is harmful to the body politic of which it forms a part. It may be incidentally mentioned at this point that on this basis a standard of values can be established in ethical matters—an impossible matter if the ethical standard is one of motives or happiness. The action of an educated man who can foresee future results is of more value than that of an ignorant man ruled by a few unbending motives.

The struggle for existence is a secondary law, being subordinate and subservient to the law of social service. The social service of the parts is improved by the betterment of the parts. Hence the worth of personality and individuality; hence the struggle for freedom in history. Self-preservation, self-control, and the perfection of one's own personality are duties, and imperative duties at that, but not categorical imperatives. "The perfection of one's powers" is, after all, only a means of obeying the categorical imperative of social service. It is here that we find the supreme court of appeal, from which there is no recourse. It may also be well to point out that from the biological and sociological standpoint it is not so much a question of the survival of individuals as a question of the survival of the best combination of parts—a much wider view.

This grounding of the social in the universal phenomena of the division of labor throws a strong light on certain prevalent theories as to the nature of sociology. One of the most prominent characteristics of this division of labor is the *differentiation of parts*. Integration of parts means the connected play of these parts, so that if

one functions the others are affected. Differentiation from other organs means individuality and difference; integration is not necessarily an interaction of similar parts, but rather an interaction of the different parts. The phenomena of integration or sociality are therefore inadequately described as "a consciousness of kind, a knowledge of resemblances, or a knowledge of like-mindedness" (Giddings).

Social life is mirrored in a football game. Each player has his function; each player thinks and acts his separate part. The signal given, the ball is snapped, each man leaps to his place, the fake pass is made, the proper interference aids the man who makes the run down the side lines, and the touch-down is made to the cheering of enthusiastic partisans. *Each man acts, I say, his part, and the element they have in common is the goal.* The common aim—the success of the team and the winning of the game—does not necessarily mean a common or similar method of action. Solidarity does not of necessity mean similarity, nor does community life mean common thoughts and actions. Nor in adult society, the training for which is the rational ground for play, do we find the process materially different. The material of social organization is not consciousness of kind, nor is it mainly such. The action of the mob, to which reference is so lovingly made by certain sociologists, is generally an instance in which the welfare of the whole is lost sight of, in which the single person becomes a unit in an *aggregation*, and in which there is a general return to the homogeneity of primitive conditions. The striking thing about a mob is not its social but its unsocial character. With the dispersion of the mob there begins again the process of differentiation and integration—true sociality. Of certain pigeons it is reported that they become extraordinarily stupid and incautious as soon as they become a part of great numbers in flight, but that they become wary, intelligent, and cautious when they are alone. Identification of the individual with the collective mass reduces it to the average level and causes temporary atrophy of certain more highly specialized qualities. The same phenomena are often observable in men and women who take refuge from their doubts and

uncertainties in the infallible doctrines of the Roman Catholic Church. Socialization is specialization, provided such specialization implies articulation and integration.

Consciousness of kind is characteristic of the lowest stages of society, and indicates a low level in a more highly evolved society. The struggle for existence implies a struggle of conflicting interests, different schools of thought and action. It is also a biological truth that the struggle is greatest between the members of the same species. It is also not necessary that each individual partner should be conscious of the common goal, provided his conduct tends that way. His motives may be wrong, but his conduct must be right. Correct motives provide, however, some guarantee of persistency of conduct. The individual lives for himself, but in so doing must serve others. Selfishness necessarily generates altruism. The chalk cliffs of the infusoria are the result of the individualistic action of each of the infusoria, the infusorium being typical of egoism.[1]

Baldwin discriminates between the substance, content, stuff, or material of society, and the functional method or process of organization of the social material. He describes the social substance or content as follows: "The matter of social organization consists of thoughts; by which is meant all sorts of intellectual states, such as imagination, knowledge and informations." This "matter," he thinks, is found only in social groups, which alone, therefore, can be called societies. Animal communities he would call "companies." The functional method or process of organization of the social material he finds in the process of imitation which is subjectively contained in the "dialectic of personal growth."

It is evident that the "substance, content, stuff, or material" of society is not the consciousness of kind, as Giddings affirms; neither can it be said that the functional method or process of organization of the social material is mainly a process of imitation, as Baldwin asserts. The process is rather that of division of labor, using that term to indicate both the process of differentiation and integration. The transmission of the social heritage, the introduction of the

(1) Ihering, *Der Zweck im Recht*, 3d ed. (Leipzig, 1893), p. 467.

young into adult social ways, may and does involve a large amount of imitation, but even there, again, it should be remarked that imitation is but one subdivision of the larger process of suggestion. Suggestion may be one of the methods by which the young acquire social ways, but it does not therefore rise to the supreme rank and importance of the social way itself. Again, imitation, and in a still larger way suggestion in all its forms, is one method of social service, as, for instance, in the influence exerted by leaders, reformers, and their like. It is, however, not to be confounded with the larger and more fundamental process of division of labor. It is deceptively epigrammatic and quite inadequate, to say with Tarde, "*Socialité, c'est l'imitativité.*"[1]

The most useful *variation* tends to survive, and hence Bailey's term, "the survival of the unlike." Variation is one of the most important processes of nature, for on this process are built the innumerable possibilities of the division of labor. Darwin's problem was, of course, the origin of differences. Linnæus, if he were still to pursue his plan of an inventory of nature as a species of natural bookkeeping, would be appalled at the number of species. Instead of the very modest forty thousand species comprising the sum total of all living species as computed by Biberg, writing in 1749 in Linnæus's *Amoenitates Academicae*, Riley concludes that "to say that there are ten million species of insects in the world would be, in my judgment, a moderate estimate." The differentiation process is proceeding as rapidly as at any period in time past; in fact, the strong probability is that it is increasing more rapidly. That organism is likely to spread most rapidly which differs most widely from all its fellows, because the field is free of competitors and there is the least impediment to its progress. This principle has been called by Darwin the divergence of character. A new character, or a new combination of characters, in any organism may tend to give such an organism an immense advantage because of the monopoly-privileges it enjoys. Freedom and liberty is the toleration of differences, affording a chance for natural or acquired aptitudes. A variation is

(1) Tarde, *Les lois de l'imitation*, p. 75.

generally useful because it accomplishes something new, something which the homogeneous mass could not do before the variation occurred. Progress is generally such differential interstitial growth.

Differentiation, however, is not invariably the open sesame to success. The secret of success lies in the degree of adaptation, and success, it may again be repeated, must be interpreted in terms of survival. We commonly say that when certain plants are transferred northward they tend to degenerate by becoming dwarfed and by losing some of their highly developed specialties. They have a tendency, like old varieties of plants, to assume some primitive or inferior type. Degeneracy or deterioration is, however, a relative term, return to a simpler form, *i. e.*, a decrease in the differentiation of the plant, being often the successful means employed to secure a further lease of life. In the same way with human beings, poverty often places an embargo on differentiation. The highly developed individual, stricken with poverty, must needs forego the satisfaction of many tastes, and revert to a more common and primitive type.

The utility of differentiation is, however, manifest on all sides. Death entered into the world with all its blessings, the old undying types giving place to the possibility of an ever-increasing variation. Sex entered, according to Weismann, and increased the number of combinations and variations. Plants become annuals and biennials from a perennial condition. Changes in the plant and animal world meet the changes of the seasons, the temperature, the food supply, and the changing demands of the rest of the plant and animal world. Bailey suggestively remarks that the development of life took two divergent lines—that of the circular arrangement of parts and that of bilateralism. The first line, developing in obedience to a peripheral or rotate type of organization, ends in the echinoderms and some of the mollusks. This type reached its zenith and, according to Cope, has left no line of descent. The progressive and regnant type of animal life appeared in the vermes, or true worms, forms which are characterized by a two-sided or bilateral, and therefore more or less longitudinal, structure. By this means greater differentiation was made possible. A cephalic or head-forming

evolution resulted from the bilateralism, and a specialization of the senses and central nervous system without parallel ensued. The extensive specialization of the sense and motor organs of the body, although marvelous, is carried to still greater limits by means of extra-organic instruments. The evolution of organs is continued in technic.[1]

The differentiation being given, integration or organization becomes possible. The more absolutely alike the parts may be, the less likelihood there will be of a superior organization. The Quakers, for instance, possess little capacity for political organization, because of the uniformity of the individuals of that persuasion. The benefits accruing from division of labor are lost, viz., the avoidance of waste effort, the increase in amount of work performed, the improvement in quality, the adaptation of the work to the natural aptitude of the workers, the greater security of the species or society, etc. Thus integration may not necessarily mean the introduction of a caste or military system; it may not follow as a logical consequence that there is an *Ueber-* and *Untereinander*, but it does imply a *Miteinander*. It may not necessitate equality or similarity of parts, but it does imply efficiency of service of the parts, which efficiency is generally in proportion to the difference in the parts. The survival of the unlike defines the fittest to be the unlike, and if, as Roux says, there is a *Kampf der Theile*, it is, we may add, *im Interesse des Ganzen*. Every man is his brother's keeper in the sense that each specialist must needs be supported by other specialists.[2] Organizations, institutions, culture, and civilization must be defined, not as products of interaction, but as such and such interactions of different parts.

(1) Otto Wiener, *Die Erweiterung unserer Sinne*, Leipzig, 1900.

(2) Hegel's statement that he who pursues a special occupation or profession does not lower himself, but only thus becomes *ein rechter Mensch*, has much truth in it.

THE LAW OF FUTURE SPECIFIC AND SOCIAL EFFICIENCY[1]

By Arthur Allin.

The social significance of education is manifest when looked at in the light of historical development. Upon careful consideration it will be seen that the educational process, far from being a temporary arrangement for present social needs, is the important factor in a general law of universal application. This law is the Law of Future Specific[2] and Social Efficiency.

In the preceding discussion it became evident that sociality has its foundations laid deep and broad in the organic life of the species, that is, in instinct, and that in the realm of the social or extra-organic the principle of sociality is if anything intensified as observable in the phenomena of the division of labor. It was seen to be not so much a question of "ought" and "should" as a question of "is" and "was." In other words it is a matter of historical accuracy as to what kind of conduct has survived and as to what kind of conduct will be most serviceable in the future in order to insure survival.

Based on the calculus of survival then the biologist and sociologist must grant that parental conduct taken in its wider, larger meaning is the "open sesame" of success, the *sine qua non* of survival, the essential condition of the life process. Propagation and providence, according to these two standards, are the nations judged in the *Weltgericht.* Goethe says with some truth:

"*Warum treibt sich das Volk so und schreit? Es will sich ernaehren,*
Kinder zeugen, und die naehren so gut es vermag.
* * * * * * * * *
Weiter bringt es kein Mensch, stell' er sich wie er auch will."

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(1) By the courtesy of the Editors reprinted from the Journal of Pedagogy, Dec., 1902.
(2) The term *specific* as used in this connection refers obviously to *species.*

The more this question is studied the more does the problem of the paramountcy of the species grow in importance. All other instincts even that of self-preservation, much lauded as it has been in popular opinion and in the deeper currents of biological and ethical discussion, are seen to play a subservient rôle.[1] The many attempts which have been made by venturesome thinkers to reduce all instincts and social acts to a fundamental instinct, viz., sexual love, have had more regard to the dramatic and striking facts of life than to the more essential underlying laws. Sexual love is a means not an end. It is instrumental in securing the propagation of the race. The phrase, "and they lived happy ever after," is a very light and airy way of saying that the goal of life has been reached, the perpetuation and better education of the species, parental care. The Roman mind, so truly and profoundly great in matters of law and government, summed up in a few words the social aspect of the most fundamental of organic laws—*salus reipublicæ esto suprema lex.*

Weismann's argument in reference to the duration of life and the problem of old age, decay and death may be applied to other problems as well, and to this problem of future specific and social efficiency especially. Writers previous to Weismann had tried to show that the age of an animal is determined by its size. This, however, cannot be the main determining agency because instances may be cited in opposition to such a theory, such as the fact that the pike and carp live as long as the elephant (200 years). Neither can the complexity of structure and function be regarded as the chief cause of length of life. Here the argument is a physiological one: the length of life being determined by the rate at which the animal lives, the rapidity with which assimilation and the other vital

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(1) Geddes and Thompson, The Evolution of Sex, Chap. XX; Laws of Multiplication, Chap. XXI; The Reproductive Factors in Evolution. "Physiologists and evolutionists are coming to see the most complex lives, in Foster's phrase, as 'but the by-play of ovum-bearing organisms.'" Darwin's "residual explanations" of sexual selection, correlation of growth, etc., are coming to be regarded as primary laws, competition or the struggle for existence and the all-sufficiency of the individual being properly subordinated to the larger and more important needs of the species and social group.

Skin for skin, all that a man has will he give for his life—a sentiment of doubtful value as an ultimate truth, and one which the Ancient who wrote the Book of Job puts it into the mouth of the Devil.

processes take place. Animals characterized by active and restless mobility are said to possess a short life, while inertness, such as that of the Amphibia, is accompanied by relatively great length of life. These statements are, however, invalidated by the fact that birds as a general rule possess great mobility and at the same time great length of life.

The true answer to the problem seems to lie in the fact that duration of life is really dependent upon adaptation to external conditions, that its length whether longer or shorter, is governed by the needs of the species, and that it is determined by precisely the same mechanical process of regulation as that by which the structure and function of an organism are adapted to its environment. Weismann writes, "Assuming for the moment that these conclusions are valid, let us ask how the duration of life of any given species can have been determined by their means. In the first place, in regulating duration of life, the advantage to the species, and not to the individual, is alone of any importance. This must be obvious to any one who has once thoroughly thought out the process of natural selection. It is of no importance to the species whether the individual lives longer or shorter, but it is of importance that the individual should be enabled to do its work towards the maintenance of the species. This work is reproduction, or the formation of a sufficient number of new individuals to compensate the species for those which die. As soon as the individual has performed its share in this work of compensation, it ceases to be of any value to the species, it has fulfilled its duty and may die. But the individual may be of advantage to the species for a longer period if it not only produces offspring, but tends them for a longer or shorter time, either by protecting, feeding or instructing them. This last duty is not only undertaken by man, but also by animals, although to a smaller extent; for instance, birds teach their young to fly, and so on.

"We should therefore expect to find that, as a rule, life does not greatly outlast the period of reproduction except in those species which tend their young; and as a matter of fact we find that this is the case."[1]

(1) Weismann, Essays on Heredity, Vol. I., p. 10.

Of course it is not necessarily true that the individual performs an instinctive or species-preserving act with the conscious aim that the species may be benefited. The conscious motive may fail but the specific conduct may not. It is a question of service rendered and the survival of the best service, natural selection being the method whereby the end is attained, viz., a greater social efficiency in the future. "A great deal," says Ritchie,[1] "of what is often blamed as the selfishness and worldly ambition and money-grubbing of the middle class is the outcome, not of direct individual selfishness at all, but of a highly developed feeling of responsibility towards offspring." A still more advanced position, however, may be taken than that of Ritchie, since a vast number of acts may be performed for personal gratification alone which in the final count may be considered as pre-eminently social in their effects. The motive does not always indicate the worth of an act.

The governing motives and passions of the great men of history when looked at from the individual standpoint of their possessors may have been saturated with the purpose of selfish aggrandizement and with the pursuit of personal glory but it must be remembered firstly, that in at least the majority of cases these men were actually instrumental in accomplishing beneficial political, economic and other changes in the social organization of the people and secondly, that the ends they sought, let us say, for selfish reasons, were necessarily social ends expressing the will of the people and destined for their ultimate good. The men glorified themselves in glorifying the people. They were personifications of the people and as such reaped their reward. The will of the people was expressed in and by these individuals. Thus Cæsar was a benefactor of mankind in that Roman law was placed on a still firmer foundation and more widely extended through his efforts. The Code Napoleon as it exists in western Germany and many Latin countries is an evidence of the social spirit of France as manifested with splendid effect through the agency of that eminently selfish demigod, Napoleon.[2]

(1) Ritchie, Natural Rights, p. 129.
(2) Works of T. H. Green, Vol. II, pp. 439 ff.; 474.

"More than a hundred years ago," says Dr. C. A. Herrick, "the world was startled with the declaration that in international trade both sides might be gainers. We recognize this at present as true for nations, but hardly so for individuals. Gain is popularly regarded somehow as illicit; if one party to a transaction has a profit, then it is felt that the other must have a corresponding loss. A generation of business men must be trained that shall see in business neither the giving nor the taking of advantages, but instead a social service for which one may expect compensation."[1]

From a consideration of the multitude of facts adduced by Sutherland the conclusion may be drawn with some degree of certainty that *the number propagated is in inverse ratio to the amount of providence and vice versa.* Education replaces large progeny in about the same manner as the play of the young of to-day is a biological and social substitute for the rigidity and predominance of instinct of earlier animal life.

Individual provision and prevision is increased, corrected and enlarged in scope by social or state provision. Taxation is being remodeled on this basis. Municipal improvements bear the imprint of the paternal. Large projects, such as transcontinental railroads, subsidization of a merchant marine, protection of growing industries, and a multitude of others which might be mentioned, all bear witness of the fact that the nations are now doing consciously what formerly they did in a more or less haphazard fashion. Despite the fact that in former times men knew not the final end for which they labored, still the hall-mark of the paternal, of future specific and social efficiency, was on their actions.

Many sociologists manifest their alarm at the decreasing birth-rate throughout the world but due regard after all must be paid to the fact that conditions have changed. The individual is not so helplessly isolated as formerly. He is provided by tradition with the lessons of the past; he is provided with a rich store of extra-organic apparatus; he is supported on all hands by social institutions.

(1) Herrick, The Content and Educational Value of the Curriculum for a Secondary School in Commerce. Proceedings of the N. E. A. 1900, pp. 543-549.

As the weeping philosopher used to say "One man is worth ten thousand, if he be the best." We are debtors of the past with no opportunity to repay; we are executors of a trust for the future—the sole duty of man. Civilization has been defined as an accumulation of forces in and for humanity.[1] Civilization is that which has been acquired *emmagasiné* by the human race, a store of capital, resources, instruments, knowledge, ideals, laws, in the last analysis a sum total of adaptations whereby the forces of nature are utilized by man for the greater safety of the species. Youth is a period of life-endowment insurance on the reserved-bonus plan. In a civilized society each generation as it rises finds itself surrounded from the cradle, thanks to the care of preceding generations, with more light and more resources of every kind. This fund, theoretically available for every one in a supposedly democratic state, is, however, it is well known, subject to the laws of inheritance, the exigencies of family management, etc. Hence the members of the small family may acquire and accomplish more than the family of larger number. To him who hath of this world's goods shall be given but from him who hath not shall be taken even that which he hath.

This universal organic and social process explains many features of great interest in the various fields of human activity. In law we view the gradual surrender of so-called individual rights to the rights of the state, the growth of freedom, *i. e.*, the development of the individual in order that the service of the whole may be increased by the strength of the constituent parts, the subordination of individual caprice to collective wisdom as seen in representative gatherings of all kinds, the predominance of self-sacrifice in the great ethical and religious movements of the world, life insurance, city sanitation, and the many acts of public philanthropy. Rousseau's dream of the social contract was of the future and for the future rather than of the past. Dante, Goethe, Shakespeare, Paul and Christ proclaim alike *das Ewig-Weibliche*, the Eternal-Motherly—Responsibility to the future of our race.

---

(1) Mon. T. Dumont, La civilisation comme force accumulée. La Revue Scientifique, 22 Juin, 1872. p. 1222. Dumont, however, makes a mistake in including under civilisation the inheritance of instincts. These are but the organic conditions of civilisation.

Der Uebermensch, the Beyond-man, as the Child of the Future, is the goal and sanction of social effort. The "law of nature" is not behind but ahead of us. The wild cry of Rousseau for equality is re-echoed by the saner demand of modern democracy for equality of opportunity — for the young. In securing the greatest happiness or rather welfare of the greatest number, we now see that the "greatest number" refers not so much to the majority now living as to that still greater majority — the unborn. Society is not composed of those now living, it represents the living as the servants of posterity. "Society," Burke declared, "is a partnership not only between those who are living, but between those who are dead, and those who are to be born."[1] And yet on the other hand we see such writers as Huxley, Spencer and others so thoroughly saturated with the idea of competition, *laissez-faire*, and struggle for existence that they lose sight, very largely, if not wholly, of the great racial struggle for future social efficiency. Huxley, in fact, pits the principles of self-sacrifice against the "cosmic process" of struggle for existence[2] and Herbert Spencer regards the ideal of the future as a state in which the interests of the individual shall become harmonized and identical with those of society.[3] "From the sociological point of view," he says, "ethics becomes nothing else than a definite account of the forms of conduct that are fitted to the associated state, in such wise that the lives of each and all may be the greatest possible, alike in length and breadth."[4]

The importance of this process is rightly estimated by Benjamin Kidd when he says, "What we see is that the highest manifestations must be drawn into the vortex of this supreme conflict. In it we stand at the very pivot of the evolutionary process in human history. The whole content of systems of thought, of philosophy, of morality, of ethics, and of religion, must in time be caught into it. It is in

(1) Quoted by Kidd, Western Civilization, p. 123.

(2) "Let us understand, once for all, that the ethical progress of society depends, not on imitating the cosmic process, still less running away from it, but in combating it." Huxley, Evolution and Ethics, Romanes Lecture. 1893.

(3) Herbert Spencer. Data of Ethics, Chap. VIII.

(4) Ibid., p. 133.

the resulting demiurgic stress that rival systems of society will be unconsciously pitted against each other; that nations, and peoples, and great types of civilization will meet, and clash, and have their principles tested. And it is respect of the controlling principle of the conflict—the degree of efficiency of the subordination of the present to the future—that Natural Selection will continue to discriminate between the living and the dead as the progress of the world continues."[1]

(1) Kidd. Principles of Western Civilization, London, 1902, p. 154.

Volume I Number 4

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On November 24, 1903, occurred the death of Professor Arthur Allin, Ph. D. Dr. Allin was the principal editor of the University of Colorado Studies and it was largely through his influence and interest that the publication was started.

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NOTES ON THE p-DISCRIMINANT OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

By Arnold Emch

1. In a fundamental memoir[1] Darboux has proved that in general the resultant $g(x, y)=0$ obtained by the elimination of $p=\frac{dy}{dx}$ between

$$\phi\ (x, y, p)=0 \tag{1}$$

and

$$\frac{\delta\phi}{\delta p}=0, \tag{2}$$

where ϕ is a polynomial in x, y, p, represents in general the locus of the cusps of the integral curves.[2]

To judge from most text-books on differential equations, Darboux's important results seem to be unknown to the majority of writers. The chapters on singular solutions are usually based on Cayley's paper: "On the theory of the singular solutions of differential equations of the first order," which first appeared in the Messenger of Mathematics, Vol. II (1873), pp. 6-12. (Also in Collected Mathematical Papers, Vol. VIII, pp. 529-534). It will be noticed that Darboux's and Cayley's investigations were published in the same year. Darboux, however, presented his results to the Société Philomatique in the session of November 23, 1872. Cayley did not discover that the p-discriminant, in general, represents the cusp-locus. Picard in Vol. III, pp. 44-52, of his Traité d' Analyse gives an elegant analytical proof for the cusp-locus.[3]

---

(1) Sur Les Solutions Singulières Des Equations Aux Derivées Ordinaires Du Premier Ordre. Bulletin des Sciences Mathématiques et Astronomiques, Vol. IV, pp. 158-176 (1873).

(2) In what follows δ is used as a partial differentiation sign.

(3) See also Dr. Schlesinger's Einführung in die Theorie der Differentialgleichungen, pp. 257-270, (Sammlung Schubert).

In what follows next I shall give a direct analytic proof for the dualistic relation between the resultants of the equations

$$\phi(x, y, p)=0, \tag{3}$$

$$\frac{\delta\phi}{\delta x}+p\frac{\delta\phi}{\delta y}=0, \tag{4}$$

and of

$$\phi(x, y, p)=0, \tag{5}$$

$$\frac{\delta\phi}{\delta p}=0. \tag{6}$$

2.[1] If in

$$\phi(x, y, p)=0,$$

x and y are replaced by the co-ordinates of a point in the (x, y) plane and (1) is solved with respect to p, then these values of p define the tangents of the integral curves through this point. But instead of considering the integral curves through a given point we may ask for the integral curves which touch a given straight line. In other words, every differential equation may be interpreted geometrically in a twofold manner, according to the principle of duality.

Now, *the locus of the points on those straight lines at which two points of tangency of integral curves with these lines coincide, is given by the resultant of*

$$\phi\ (x, y, p)=0,$$

and

$$\frac{\delta\phi}{\delta x}+p\frac{\delta\phi}{\delta y}=0,$$

and represents the locus of the points of inflexion of the integral curves. (See Darboux, loc. cit.).

To get the dualistic interpretation of this theorem we make a transformation by reciprocal radï of the differential equation and its sytem of integal curves with respect to the circle

$$x^2+y^2=1.$$

[1] This section has been presented to the American Mathematical Society, at the annual meeting in Boston, August, 1903. See Bulletin of the American Mathematical Society, 2nd Series, Vol. X, pp. 137-139, Dec. 1903.

Designating the transformed point of (x, y) by (x_1, y_1) we have[1] the relations:

$$\left.\begin{aligned} x &= \frac{-p_1}{y_1 - x_1 p_1}, \\ y &= \frac{1}{y_1 - x_1 p_1}, \\ p &= \frac{x_1}{y_1}, \end{aligned}\right\} \qquad (7)$$

$$\left(p = \frac{dy}{dx}, p_1 = \frac{dy_1}{dx_1}\right)$$

by which (1) is transformed into another linear differential equation of the same order

$$\phi_1 (x_1, y_1, p_1) = 0. \qquad (8)$$

By this transformation the equation

$$\frac{\delta \phi}{\delta x} + p \frac{\delta \phi}{\delta y} = 0 \text{ becomes}$$

$$\frac{\delta \phi}{\delta \phi_1} \cdot \frac{\delta \phi_1}{\delta x} + p \frac{\delta \phi}{\delta \phi_1} \cdot \frac{\delta \phi_1}{\delta y} = 0,$$

or

$$\frac{\delta \phi}{\delta \phi_1} \left\{ \left(\frac{\delta \phi_1}{\delta x_1} \cdot \frac{\delta x_1}{\delta x} + \frac{\delta \phi_1}{\delta y_1} \cdot \frac{\delta y_1}{\delta x} + \frac{\delta \phi_1}{\delta p_1} \cdot \frac{\delta p_1}{\delta x} \right) + p \left(\frac{\delta \phi_1}{\delta x_1} \cdot \frac{\delta x_1}{\delta y} + \frac{\delta \phi_1}{\delta y_1} \cdot \frac{\delta y_1}{\delta y} + \frac{\delta \phi_1}{\delta p_1} \cdot \frac{\delta p_1}{\delta y} \right) \right\} = 0. \qquad (9)$$

But, according to (7)

$$\frac{\delta x_1}{\delta x} = -\frac{p^2}{(y - xp)^2}, \quad \frac{\delta y_1}{\delta x} = \frac{p}{(y - xp)^2}, \quad \frac{\delta p_1}{\delta x} = -\frac{1}{y},$$

$$\frac{\delta p_1}{\delta y} = \frac{x}{y^2}, \quad \frac{\delta y_1}{\delta y} = -\frac{1}{(y - xp)^2}, \quad \frac{\delta x_1}{\delta y} = \frac{p}{(y - xp)^2}.$$

Substituting these values in (9), we get from (9)

$$\frac{\delta \phi}{\delta \phi_1} \cdot \frac{\delta \phi_1}{\delta p_1} \left(-\frac{1}{y} + p \frac{x}{y^2} \right) = 0,$$

[1] S. Lie, Berührungstransformationen, Vol. I, p. 23.

but as $\frac{\delta \phi}{\delta \phi_1}$ and $-\frac{1}{y'} + p\frac{x}{y^2}$ do not vanish, generally, the condition reduces to

$$\frac{\delta \phi_1}{\delta y_1} = 0. \tag{10}$$

We have therefore the theorem, *to the simultaneous equations*

$$\phi\ (x, y, p) = 0, \quad \frac{\delta \phi}{\delta x} + p\frac{\delta \phi}{\delta y}$$

correspond dualistically the equations

$$\phi_1(x_1, y_1, p_1) = 0, \quad \frac{\delta \phi_1}{\delta p_1} = 0.$$

But to a point of inflexion corresponds dualistically a cusp, hence Darboux's theorem:

The equation resulting from the elimination of p between $\phi_1 = 0$ and $\frac{\delta \phi_1}{\delta p_1} = 0$, or the p_1-discriminant of the differential equation $\phi_1 = 0$ represents in general the locus of the cusps of the integral curves.

3. At the end of his paper, referred to above, Cayley states: "By what precedes, it appears that the p-discriminant locus is made up of the envelope locus, cuspidal locus, and the tac-locus; as I infer, each of them once." (It ought to be twice for the tac-locus). This proposition and a similar one concerning the c-discriminant of a system of curves, $f(x, y, c) = 0$, were given without proof by Cayley. J. M. Hill proved them in an elaborate article published in the Proceedings of the London Mathematical Society[1] of 1889.

That the tac-locus, if it exists, occurs twice in the p-discriminant, was proved by Darboux in his fundamental memoir of 1872. As an example Darboux assumes the system of integral curves of a special differential equation in the form

$$\lambda^2 A + \lambda B + C = 0, \tag{11}$$

where A, B, and C are functions of x and y, and λ is the constant of

[1] On the c- and p-discriminant of Integrable Differential Equations of the First Order, Vol. XIX, pp. 561-589.

integration. The differential equation belonging to (11) has the form

$$p^2A_1+pB_1+C_1=0, \tag{12}$$

and the p-discriminant is

$$B_1^2-4A_1C_1=0. \tag{13}$$

Determining A_1, B_1, C_1 from (11) in the usual manner and substituting them in (13) the result is

$$(B^2-4AC)\begin{vmatrix} A & B & C \\ \frac{\delta A}{\delta x} & \frac{\delta B}{\delta x} & \frac{\delta C}{\delta x} \\ \frac{\delta A}{\delta y} & \frac{\delta B}{\delta y} & \frac{\delta C}{\delta y} \end{vmatrix}^2=0 \tag{14}$$

If $B^2-4AC=0$, then for every point of this locus the two λ's, i. e., the two integral curves through this point coincide. In other words $B^2-4AC=0$ is the envelope locus of (11).

For a point of the locus represented by the determinant, the two λ's in (11) are different, i. e., at such a point two distinct integral curves are tangent to each other. The locus represented by the determinant in (14) is therefore the tac-locus, and it clearly appears twice in the p-discriminant (14).

4. This critical and historic account on the p-discriminant would not be complete without a few remarks concerning the seeming discrepancy between Darboux's and Cayley's results.

Darboux does not assume any knowledge on the system of integral curves of a given differential equation, i. e., he considers the most general equations without reference to their possible origin. Cayley however considers only such equations, which traditionally are formed from a given system of curves, $f(x, y, c)=0$. In other words he assumes that every differential equation admits of such a specific system, having an envelope generally.

The fact is, as Picard states, loc. cit., that to an *arbitrary* differential equation corresponds a family of curves which from the point

of view of the theory of envelopes have *special* properties, while to an *arbitrary* system of curves corresponds a differential equation which from the point of view of the theory of differential equations is *special.*

SEPTEMBER, 1903.

NEWTON'S FIVE TYPES OF PLANE CUBICS OBTAINED BY THE STEINERIAN TRANSFORMATION

By Arnold Emch

Introduction.

1. The transformation by which plane cubics may be studied in a very simple and elegant manner was investigated by Steiner in his "Systematische Entwickelungen, etc."[1]

Without giving it Steiner's original form, the transformation may be defined as follows: Let $P+\lambda Q=0$ represent a pencil of conics whose fundamental points $A_1A_2A_3A_4$ may be all real, or all imaginary, or two real and two conjugate imaginary. Now it is known that the polars of any point X with respect to the conics of the given pencil are concurrent at a point X'. Thus, in general, to every point X of the plane of the pencil corresponds a point X'. This correspondence is called the *Steinerian transformation* and is evidently involutoric.

In Fig. 1 the fundamental quadruple has been assumed entirely real, in such a manner that $A_1A_2A_3$ is an equilateral triangle and A_4 the point of concurrence of its altitudes. This special assumption has no bearing upon the subsequent reasoning. In this case the diagonal points $B_1B_2B_3$ of the quadruple are the foot-points of the altitudes.

To a point Q of a side $\overline{A_i\,A_j}$ corresponds the harmonic point Q' on this side with respect to A_i and A_j, so that $(QQ'A_iA_j)=-1$. To a diagonal point B_1 corresponds every point of the opposite diagonal $\overline{B_2\,B_3}$. The points A are self-corresponding. Otherwise the correspondence in the whole plane is uniform.

Assuming $\overline{A_1A_4}$ as the x-axis and the perpendicular through

(1) See also his "Werke," Vol. I, pp. 407-421 and M. Disteli: Die Metrik der circularen Curven dritter Ordnung im Zusammenhang mit geometrischen Lehrsätzen Jakob Steiners. Also Poncelet: Traité des propriétés projectives des figures, 1 ed., 1822, p. 198, and Transon (projection gauche, Nouv. Ann. II, 4 & 5).

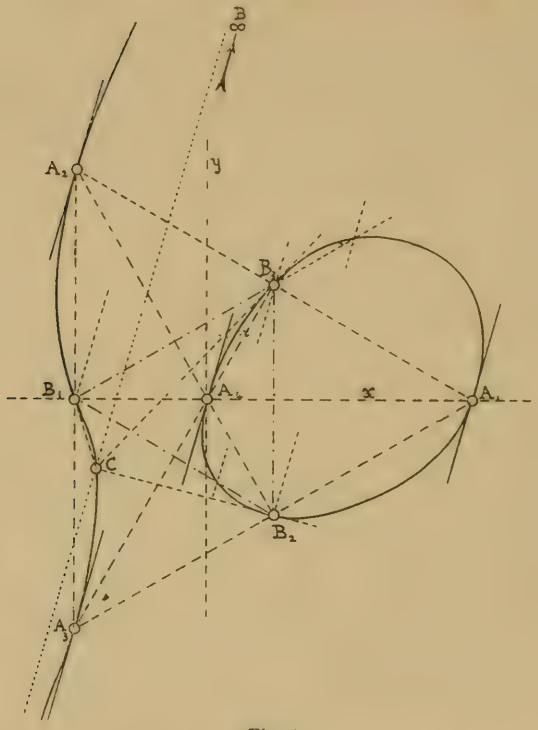

Fig. 1

A_4 as the y-axis; furthermore $A_1A_4=A_2A_4=A_3A_4=1$, it can easily be verified that to a point (x,y) corresponds in the Steinerian transformation a point (x', y') so that

$$x' = \frac{2(x^2-y^2)+x}{4(x^2+y^2)-1},$$

$$y' = \frac{y-4xy}{4(x^2+y^2)-1},$$

and conversely

$$x = \frac{2(x'^2-y'^2)+x'}{4(x'^2+y'^2)-1},$$

$$y = \frac{y'-4x'y'}{4(x'^2+y'^2)-1}.$$

To the line at infinitely ($x=\infty$, $y=\infty$) corresponds the circle

$$x'^2+y'^2=\tfrac{1}{4},$$

i. e., the circle passing through B_1 B_2 B_3.

In order to get the analytical expression for the most general Steinerian transformation we simply have to apply a general collineation to the above expressions, since by a collineation a quadrilateral may be transformed into any other quadrilateral.

From this it is easily seen that to a curve of the n^{th} order corresponds a curve of the $2n^{\text{th}}$ order. To a straight line corresponds a conic through B_1 B_2 B_3. To the line at infinity corresponds also a conic through B_1 B_2 B_3 which moreover cuts the sides A_i A_j at their middle-points. Thus, nine points of this conic are known at the outset.

2. In the Steinerian transformation cubics appear by the following considerations: To a straight line g corresponds a conic G. (I shall use $g=0$, $G=0$ as abbreviated equations of these lines) which may cut g in two real, or two imaginary points X, X′, which correspond to each other in the transformation. The same straight line cuts the pencil $P+\lambda Q=0$ through the fundamental quadruple in an involution whose double-points coincide with X and X′. Taking a pencil of straight lines, $g+\lambda h=0$, through a point B, then on every line of this pencil there are two points X and X′. The locus of these points is a cubic through A_1 A_2 A_3 A_4, B_1 B_2 B_3, B and its corresponding C. The lines from B to the fundamental points are tangents to the cubic. Transforming this cubic by the same transformation it is found that it is transformed into itself. If the fundamental quadrilateral is such that B_1 B_2 B_3 are the foot-points of the altitudes of A_1 A_2 A_3 and if B is infinitely distant, the cubic associated with B is circular. In what follows I shall, for the convenience of their construction, consider only such cubics. The results obtained thereby may be immediately extended to the general case by collineations.

Designating the slope of the direction of B by m, the equation of the cubic in Fig. 1 becomes

$$mx-y+\frac{y-4xy}{4(x^2+y^2)-1}-m\frac{2(x^2-y^2)+x}{4(x^2+y^2)-1}=0.$$

By a collineation which transforms the quadruple of Fig. 1 into another orthogonal quadruple, i. e., a quadruple in which $B_1B_2B_3$ are the foot-points of the altitudes, this equation assumes the form

$$(ax+\beta y)(x^2+y^2)+ax^2+2bxy+cy^2+2dx+2ey+f=0,$$

which is the equation of the general circular cubic. Again, by collineations any cubic may be transformed into a form whose equation has the form

$$y^2=a(x-e_1)(x-e_2)(x-e_3),$$

with $e_1 \geqq e_2 \geqq e_3$. According to the values of e_1, e_2, e_3, the cubic may belong to one of five classes as first established by Newton.[1]

I. The cubic serpentine with oval (parabola campaniformis cum ovali).

The e's are all different from each other and all real.

II. The cubic serpentine (parabola pura).

e_1 is real, and e_2 and e_3 are conjugate imaginary.

III. The cubic serpentine with isolated point (parabola punctata).

$e_1=e_2$ different from e_3 and all real.

IV. The nodal cubic (parabola nodata).

e_1 different from $e_2=e_3$ and all real.

V. The cuspidal cubic (parabola cuspidata).

$e_1=e_2=e_3$ and all real.

This somewhat lengthy introduction will be sufficient for the understanding of the constructions of the five types as they follow from the Steinerian transformation.

I. The Cubic Serpentine with Oval.

This cubic is obtained when all four points of the fundamental quadruple are either real, or imaginary. As the case of four real

[1] Enumeratio linearum tertii ordinis (Londini, 1706).

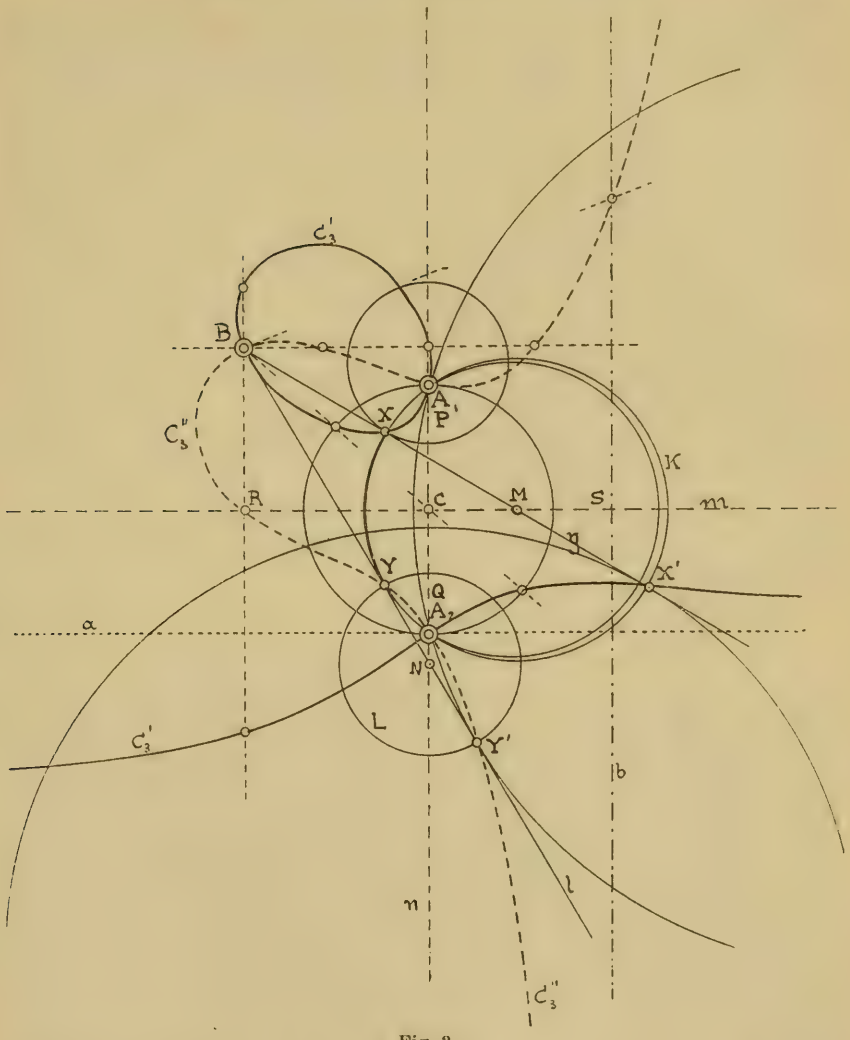

Fig. 2

points is illustrated by Fig. 1, I shall now assume an entirely imaginary quadruple which is determined by the imaginary fundamental points of an elliptic coaxial system of circles and the circular points at infinity through which it also passes. Let P and Q be the limiting points of the system, Fig. 2. On every ray g through an arbi-

trary fixed point B the circles of this system cut out an involution of points whose double-points X and X′ are two points of the circular cubic associated with the point B in the Steinerian transformation of the given imaginary quadruple. The points X and X′ are also the points of tangency of g with two circles of the given coaxial system. Hence, according to a well known construction, the points X and X′ are obtained by finding the point of intersection M of g with m, the line joining the finite imaginary points of the quadruple. With M as a center pass a circle K through P and Q which will cut g in the required points. From the figure it is seen that the two points of the cubic on a ray through B are equally distant from m. Hence, taking a ray through B parallel to m, the point at infinity corresponding to Q will be in a line a through P parallel to m. In other words, the line a is the real asymptote of the cubic. Considering the pencil of circles through P and Q, the same circular cubic is also produced by this pencil and the pencil of corresponding diameters through B.

II. The Cubic Serpentine.

This curve is produced by assuming two separate real and two conjugate imaginary points as the fundamental quadruple. In Fig. 2 let A_1 and A_2 be the real points and the circular points of the pencil of circles through A_1 and A_2 the imaginary points. To find the points Y and Y′ where a ray l through B cuts the cubic, let l cut n at N. With N as a center construct the circle L orthogomal to the pencil of circles through A_1 and A_2. The circle cuts l in the required points Y and Y′. This cubic appears, again, plainly as the product of a pencil of circles and a pencil of diameters through B. Two points Y and Y′ on a ray through B are always equally distant from n. To R corresponds the infinitely distant point of the cubic; consequently the asymptote b is parallel to n and its distance SC from m is equal to RC.

III. The Cubic Serpentine with Isolated Point.

The quadruple consists of two distinct points A_1A_3 and two coincident points A_2A_4. It is assumed that the direction of the line joining A_2 with A_4, in the limit, i. e., as they become coincident, cuts

A_1A_3 at B_2. B_1 and B_2 coincide with A_2A_4, Fig. 3. In the Steinerian transformation we find the point C corresponding to a point B, by joining B to B_1, B_2, B_3 and constructing the fourth harmonic rays

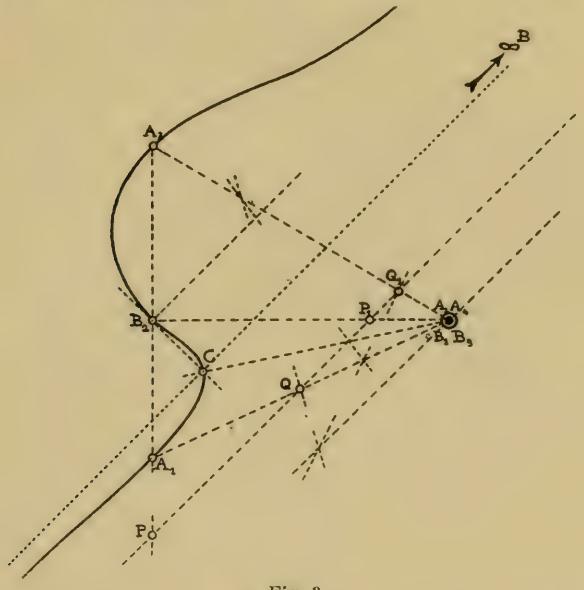

Fig. 3

to these joining lines with respect to the pairs of sides of the quadruple through the points B. The three fourth harmonic rays concur at the required point C. In our case the rays B_1C and B_3C coincide, as can easily be seen by passing over to the limit. As in the general case of a real quadruple, they cut the fourth harmonic ray through B_2 at C, the point through which the asymptote passes. The pencil of conics through the quadruple cuts every ray through B to the left of A_3 and the right of A_1 in elliptic involutions, and only the rays between A_1 and A_3 contain hyperbolic involutions. The only branch of the cubic is therefore contained between two lines through A_1 and A_3 parallel to the direction of B. The ray through A_2A_4 carries a parabolic involution and A_2A_4 represents an isolated point of the cubic.

IV. The Nodal Cubic.

Assuming in the fundamental quadruple A_1 and A_4 real and co-

incident and A_2, A_3 conjugate imaginary, a cubic with a double-point, or node, at A_1A_4 arises. In Fig. 4, a vertical line through A_1A_4 represents the limiting direction of the line joining the two

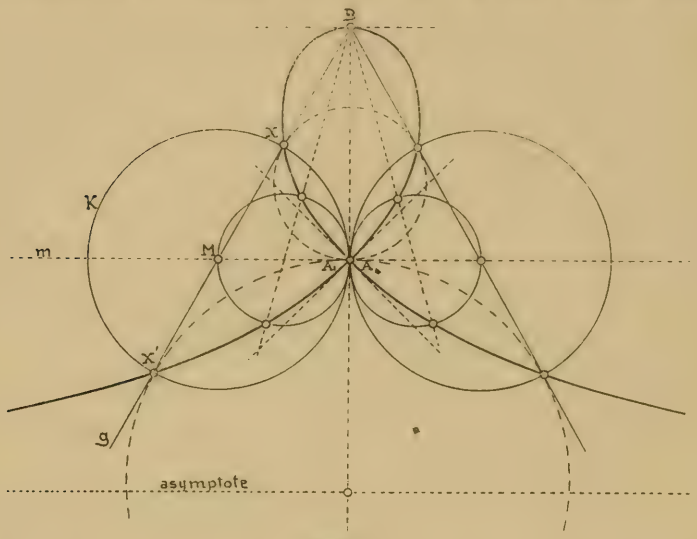

Fig. 4

points. As conics of the pencil through the fundamental quadruple take the pencil of circles tangent to each other at A_1A_4 and to the vertical line. A_2 and A_3 are then represented by the circular points at infinity. To construct the cubic associated with an arbitrary point B, draw rays through B. On each of these rays the pencil of circles cuts out an involution whose double-points are points of the cubic. These points are also the points of tangency of circles of the pencil. Hence, to find the points where a ray g through B cuts the cubic, take the point M where g cuts m as a center of a circle K passing through A_1A_4. K cuts g in the required points X and X′. From this it is seen that this cubic is also the product of a pencil of circles with coincident limiting points and a pencil of diameters through B. As X and X′ are equally distant from m, the asymtote is parallel to m at a distance to the left of m equal to BA_1 ($BA_1 \perp m$ for the sake of symmetry).

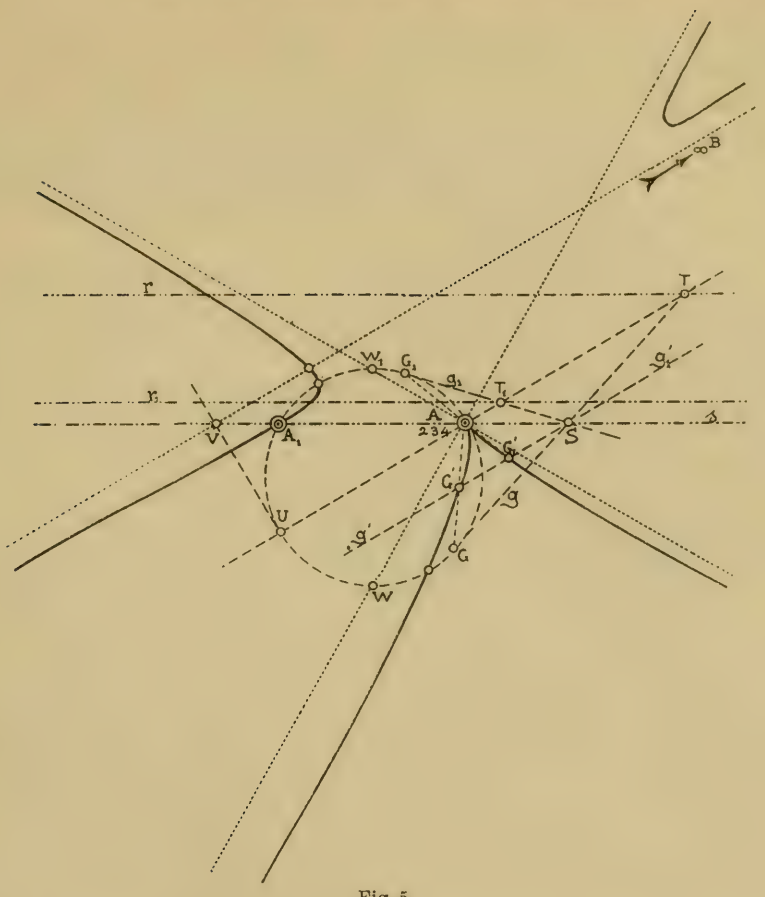

Fig. 5

V. The Cuspidal Cubic.

In this case three of the four points of a real fundamental quadruple coincide. Constructively such an arrangement can be realized by assuming as the pencil of conics a pencil through a fixed point A_1 and with its conics osculating each other at another fixed point which evidently may be considered as the representative of the three coincident points $A_2A_3A_4$.

To construct a pencil of osculating conics we may start with the fact that the picture of a circle in a perspective collineation whose

center lies on the axis of collineation and also on the given circle is a conic osculating the given circle at the center of collineation [1]. Hence, considering in Fig. 6[2] the line s, joining A_1 with the coincident remaining points, as the common axis of an infinite number of perspective collineations in which only the counter-axes[3] vary, then the pictures of a fixed circle K through $A_1A_2A_3A_4$ clearly form a pencil of osculating conics.

On every ray g' (or the identical g_1') through a fixed point B (assumed infinitely distant) this pencil cuts out an involution whose double-points are two points on the cuspidal cubic associated with B in the Steinerian transformation. These points are also the points of tangency of g' (g_1') with two conics of the pencil. For the actual construction the following simple method may be applied. Let g' intersect s at S. From S draw the two tangents g and g_1 to the circle K. Through the center of collineation (cusp) draw a line l parallel to the direction of B. Let T and T_1 be the points of intersection of l with g and g_1, and through T and T_1 draw two lines r and r_1 parallel to s. Considering r and r_1 as counter-axes of two collineations with the same axis s and the same center, then, according to the constructions of collineation, g' and g_1' are the pictures of g and g_1 in these two collineations, and the rays joining C to G and G_1 cut g' (g_1') in two points G′ and G_1' which evidently are the points of tangency with g' (g_1') of the two osculating conics corresponding to K in the two collineations (r, r_1). The line l cuts K at U; the tangent at U cuts s at V, and from the construction follows that the line through V, parallel to l, is an asymptote. In a similar manner the lines joining C to the points of tangency W and W_1 of the tangents to K, parallel to s, are the directions of the asymptotes.

By proper collineations it is not difficult to transform the five cubics constructed by means of the Steinerian transformation into Newton's five symmetrical types.

(1) Fiedler: Darstellende Geometrie, Vol. I (3rd ed.), pp. 188-190.
(2) The branch of the cubic on the upper right side has only been indicated. In the construction it fell beyond the border of the figure.
(3) See Fiedler, loc. cit., pp. 47-49.

GROUPS OF ORDER P^m, WHICH CONTAIN CYCLIC SUB-GROUPS OF ORDER P^{m-2} [1]

By L. I. Neikirk

The groups of order p^m, which contain self-conjugate cyclic sub-groups of orders p^{m-1} and p^{m-2} respectively, have been determined by Burnside, Theory of Groups of a Finite Order, pp. 75-81, and the number of groups of order p^m, which contain cyclic non self-conjugate sub-groups of order p^{m-2} has been given by Miller, Transactions of the American Mathematical Society, Vol. III, No. 4 and Vol. II, No. 3. Prof. Miller has used a method which partially depends on a special form of representation of these groups, i. e., as substitution groups.

The method of treatment used in this paper is entirely abstract in character, and in virtue of its nature, it is possible in each case to give explicitly the generational equations of these groups. They are divided into two classes, and it will be shown that these classes correspond to the two partitions (m–2, 2) and (m–2, 1, 1).

Assume an abstract group G of order p^m. G contains operators of order p^{m-2} and no operator of greater order exists in G. Let P denote one of the operators of G of order p^{m-2}. The p^2 power of every operator in G is contained in the cyclic sub-group $\{P\}$, otherwise the order of G would be greater than p^m.

The division into classes is effected by the following assumptions:

1° There is in G at least one operator Q, whose p power is not contained in $\{P\}$.

2° The p power of every operator in G is contained in $\{P\}$.

[p is assumed an odd prime and m is taken greater than 4 the

[1] Thesis presented to the Graduate Faculty of the University of Colorado for the degree of Master of Science.

groups of order p^m, $m=1, 2, 3, 4$, being given by Burnside pp. 87-88.]

CLASS I.

The group G is generated by P and Q since the operators $Q^\beta P^\alpha$ $\beta=0,1,2 \ldots p^2-1$, $\alpha=0,1,2 \ldots p^{m-2}_{-1}$, are p^m in number and are all different. We have the relation $Q^{p^2}=P^{kp^2}$. In G there is a sub-group H_1 of order p^{m-1} which contains $\{P\}$ self-conjugately, and H_1 is self-conjugate in G.

[Burnside Theory of Groups, Art. 54, p. 64.]

H_1 is generated by P and some other operator $Q^\beta P^\alpha$ of G. Then Q^β is contained in H_1 and H_1 is the sub-group $\{P, Q^p\}$ and the operations of H_1 are of the form $Q^{\beta_p}P^\alpha$; $\beta=0, 1, 2, \ldots p^{-1}$; $\alpha=0, 1, 2, \ldots p^{m-2}_{-1}$.

From this we have the two equations

$$Q^{-p}PQ^p=P^{1+kp^{m-3}} \qquad (1)^*$$

$$Q^{-1}PQ=Q^\beta pP^\alpha \qquad (2)$$

[*Burnside Theory of Groups, Art. 56.]

Determination of the sub-group H_1.

By a repeated application of (1) we obtain

$$Q^{-yp}P^xQ^{yp}=P^x[1+kp^{m-3}]^y=P^x[1+kyp^{m-3}]$$

where $m \geqq 4$

and from this it follows that

$$[Q^{yp}P^x]^s=Q^{syp}P^x\left[s+ky\frac{s(s-1)}{2}p^{m-3}\right] \qquad (3)$$

Determination of G.

It now follows from (2) and (3)

$$Q^{-p}PQ^p=Q^{\beta p}\frac{a^p-1}{a-1}P^{a^p}\left[1+\frac{k\beta}{2}\frac{a^2-1}{a-1}p^{m-3}\right]$$

$$=P^{1+kp^{m-3}}.$$

Hence

$$\beta p \frac{a^p-1}{a-1} \equiv 0 \pmod{p^2} \text{ and}$$

$$a^p\left[1+\frac{k\beta}{2}\frac{a^p-1}{a-1}p^{m-3}\right]+h\beta p\frac{a^p-1}{a-1} \equiv 1+kp^{m-3} \pmod{p^{m-2}}$$

From the last congruence

$$a^p \equiv 1 \pmod{p^2} \text{ where in } \geqq 5 \text{ and}$$
$$a \equiv 1 \pmod{p}$$

Substituting $1+ap$ for a in the second congruence we obtain after a reduction

$$\frac{(1+ap)^p-1}{ap^2}[a+h\beta]p^2 \equiv kp^{m-3} \pmod{p^{m-2}}$$

from which, making use of the fact that

$$\frac{(1+ap)^p-1}{ap^2} \equiv 1 \pmod{p}$$

we obtain

$$(a+h\beta)p^2 \equiv 0 \pmod{p^{m-3}}.$$

From these last two congruences

$$(a+h\beta)p^2 \equiv kp^{m-3} \pmod{p^{m-2}}. \tag{4}$$

Equation (2) is now replaced by

$$Q^{-1}PQ = Q^{\beta p}P^{1+ap}. \tag{5}$$

The group G is completely defined by (5) and (4) with the relation $Q^{p^2}=P^{hp^2}$. These relations may be presented in a simpler form through a replacement of operators.

From (3), (4), and (5)

$$(Q^yP^x)^s = Q^{sy}+\frac{s(s-1)}{2}xy\beta pP^x\left[s+\frac{s(s-1)}{2}ayp\right]+\frac{s(s-1)}{2}\frac{x(x-1)}{2}$$
$$k\beta yp^{m-3}+\frac{1}{2}\frac{s(s-1)(s-2)}{3}x^2y\beta kp^{m-3}+\frac{xak}{2}\left[y^2\frac{s(s-1)(2s-1)}{6}-\right.$$
$$\left.\frac{s(s-1)}{2}y\right]p^{m-3}.$$

Placing $s=p^2$ and $y=1$

$$(QP^x)^{p^2}=Q^{p^2}P^{xp^2}=P^{(x+h)p^2}$$

If x be so chosen that

$$x+k\equiv 0 \pmod{p^{m-4}}$$

QP^x will be an operator of order p^2 and may be taken in place of Q. The group G is generated by P and Q,

$$P^{p^{m-2}}=1 \quad Q^{p^2}=1.$$

As a direct consequence of the foregoing relation the groups in this class correspond to the partition (m—2, 2)
Equation (5) is replaced by

$$Q^{-1}PQ=Q^{\beta p}P^{1+\alpha p^{m-4}} \tag{6}$$

as may be seen from congruence (4).
From (6) and (3)

$$Q^{-y}P^xQ^y=Q^{xy\beta p}P^{x(1+\alpha yp^{m-4})+y\frac{x(x-1)}{2}k\beta p^{m-3}} \tag{7}$$

When α and β in (6) take all possible values consistent with the defining relations, cases of simple isomorphism between the resulting groups will arise. In order to exclude these all groups are reduced to type forms.

The groups are now sub-divided under six heads

1° α and β are both prime to p.

These groups are reduced to a single type in which α and β are each unity.

This is accomplished by replacing P^x, Q^y by P, Q where x and y are so chosen as to satisfy

$$x\beta\equiv 1 \pmod{p}$$

$$\alpha y+\frac{y(x-1)}{2}k\beta p\equiv 1 \pmod{p^2}$$

These congruences follow at once from (7) and admit of solution.

The type group is generated by the independent operators P and Q, which are subject to the relations

$$P^{p^{m-2}}=1 \quad Q^{p^2}=1$$

$$Q^{-1}PQ=Q^p P^{1+p^{m-4}}$$

By an analogous replacement of operators in the other cases we obtain the type groups

2^0 $Q^{-1}PQ=Q^pP^{1+p^{m-3}}$

3^0 $Q^{-1}PQ=Q^pP$

4^0 $Q^{-1}PQ=P^{1+p^{m-4}}$

5^0 $Q^{-1}PQ=P^{1+p^{m-3}}$

6^0 $Q^{-1}PQ=P$

The last three of the above types contain $\{P\}$ self-conjugately, while the last is the Abelian group of the type (m—2, 2).

Class II.

There is in G a sub-group H_1 of order p^{m-1} which contains $\{P\}$ self-conjugately. (Burnside, Art. 54, p. 64.)

This is generated by P and some operator Q. By the hypothesis of this class Q^p is contained in $\{P\}$. $Q^p=P^{hp}$.

All the operators of H_1 are given by

$$Q^\beta P^\alpha \quad \alpha=0, 1, 2, 3 \ldots p^{m-2}; \ \beta=0, 1, 2, \ldots p^{-1}.$$

From this

$$Q^{-1}PQ=P^{1+kp^{m-3}} \qquad (1)^*$$

[*Burnside Theory of Groups, Art. 56.]

Determination of H_1.

From (1)

$$Q^{-y}P^xQ^y=P^x[1+kp^{m-3}]y=P^x[1+kyp^{m-3}] \qquad (2)$$

where $m \geq 4$,

and from this it follows

$$[Q^y P^x]^s = Q^{sy} P^{x[s+ky\frac{s(s-1)}{2}} p^{m-3}] \qquad (3)$$

Placing s=p and y=1

$$[QP^x]^p = Q^p P^{xp} = P^{(x+h)p}.$$

If x be so chosen that

$$x+h \equiv 0 \pmod{p^{m-3}}$$

QP^x will be an operator of the order p and may be taken in place of Q.

The sub-group H_1 is generated by P and Q where

$$P p^{m-2} = 1 \qquad Q^p = 1,$$

with equation (1).

Determination of G.

G is generated by H_1 and some operator R of G which is not contained in H_1. By the hypothesis of this class

$$R^p = P^{lp}. \quad G = R^a H_1, \; a = 0, 1, 2 \; . \; . \; . \; p^{-1}.$$

Since H_1 is self-conjugate in G

$$R^{-1} P R = Q^\beta P^a \qquad (4)$$

$$R^{-1} Q R = Q^b P^{ap^{m-3}} \qquad (5)^*$$

[*Burnside, Art. 24.]

By a repeated application of (4), (5) and (3)

$$R^{-p} P R^p = Q \frac{b^p - a^p}{b-a} \beta \, P^{ap} [1 + \frac{k\beta}{2} \frac{a^p - 1}{a-1} p^{m-3}]$$

$$+ a\beta \left[\frac{p a^{p-1}}{a-b} - \frac{a^p - b^p}{(a-b)^2} \right] p^{m-3} = P.$$

Hence

$$\frac{b^p - a^p}{b-a} \beta \equiv 0 \pmod{p} \quad \text{and}$$

$$a^b [1 + \frac{k\beta}{2} \frac{a^p - 1}{a-1} p^{m-3}] + a\beta \left[p \frac{a^{p-1}}{a-b} - \frac{a^p - b^p}{(a-b)^2} \right] p^{m-3} \equiv 1 \pmod{p^{m-2}}.$$

From the last congruence

$$a^p \equiv 1 (\mathrm{mod}\ p^{m-3}) \text{ and}$$

$$a \equiv 1 (\mathrm{mod}\ p^{m-4}).$$

Equation (4) is now replaced by

$$R^{-1}PR = Q^{\beta}P^{1+ap^{m-4}}. \tag{6}$$

From (5), (6) and (3) with (2)

$$R^{-p}QR^p = P^{-1p}QP^{1p} = Q^{b^p}P^{\frac{b^p-1}{b-1}ap^{m-3}} = Q.$$

Hence

$$b^p \equiv 1 (\mathrm{mod}\ p) \text{ and}$$

$$a\frac{b^p-1}{b-1} \equiv 0 (\mathrm{mod}\ p).$$

From the first congruence

$$b = 1.$$

Equation (5) now becomes

$$R^{-1}QR = QP^{ap^{m-3}} \tag{7}$$

and equation (6) becomes

$$R^{-1}PR = Q^{\beta}P^{1+ap^{m-3}}. \tag{8}$$

This last comes by the substitution of $1+ap^{m-4}$ for a and 1 for b in the congruence determining a.

From (3), (7) and (8)

$$R^{-y}P^xR^y = Q^{\beta xy}P^x\left[1+ayp^{m-3}+a\beta\frac{y(y-1)}{2}p^{m-3}\right]$$
$$+k\beta y\frac{x(x-1)}{2}p^{m-3}, \tag{9}$$

$$R^{-y}P^xR^y = Q^xP^{axyp^{m-3}} \tag{10}$$

From (2), (9) and (10)

$$[R^zQ^yP^x]^s = R^{sz}Q^{sy+\frac{s(s-1)}{2}\beta xz}P^x\left[s+\frac{s(s-1)}{2}az\,p^{m-3}\right.$$

$$+\frac{a\beta}{2}\left\{\frac{s\,(s-1)\,(2s-1)}{6}\,z^2-\frac{s\,(s-1)}{2}\,z\right\}p^{m-3}]+k\beta\frac{s(s-1)}{2}$$

$$\frac{x(x-1)}{2}\,zp^{m-3}+\{\,x\,k\,\frac{s\,(s-1)}{2}\,y+k\,\beta x^2\,z\,\frac{1}{2}\,\frac{s(s-1)(s-2)}{3}\Big\}\,p^{m-3}$$

$$+\frac{s\,(s-1)}{2}\,a\,y\,z\,p^{m-3}. \qquad (11)$$

Placing $z=1$ $y=s=p$

$$[R\,P^x]^p=R^p\,P^{xp}=P^{(x+1)\,p}$$

If x be so chosen that

$$x+1\equiv 0\ (\mathrm{mod}\ p^{m-3})$$

RP^x is an operator of order p which may be used in place of R.

G is generated by P, Q and R

$$P\,P^{p^{m-2}}=1 \quad Q^p=1 \quad R^p=1$$

with relations (1), (7) and (8).

As a direct consequence of the above these groups correspond to the partition (m—2, 1, 1).

When the parameters in (1), (7) and (8) take all possible values consistent with our fundamental assumptions, cases of simple isomorphism between the resulting groups will arise.

The simply isomorphic groups are reduced to type forms by a transformation.

Chose a second set of generators P', Q', and R', then by a repetition of the foregoing argument

$$P'P^{m-2}=1 \quad Q'^p=1 \quad R'^p=1$$

$$Q'^{-1}P'Q'=P'^{1+k'p^{m-3}} \qquad (12)$$

$$R'^{-1}P'R'=Q'^{\beta'}P'^{1+a'p^{m-3}} \qquad (13)$$

$$R'^{-1}Q'R'=Q'P'^{a'p^{m-3}}. \qquad (14)$$

A consideration of (11) shows that

$$P'=R^zQ^yP^x \quad x\ \text{not}\equiv 0(\mathrm{mod}\ p)$$

$$Q' = R^{z'} Q^{y'} P^{x'} p^{m-3}$$

$$R' = R^{z''} Q^{y''} P^{x''} p^{m-3}.$$

The condition that the lowest power of Q′ or R′ in $\{P'\}$ shall be the p^{th} gives

either z′ or y′ not $\equiv 0 \pmod p$ and

either z″ or y″ not $\equiv 0 \pmod p$.

Let the lowest power of R′ in $\{P', Q'\}$ be $R'^{s''}$ then

$$R'^{s''} = Q'^{s'} P'^{s} p^{m-3}.$$

Substitute the values of P′, Q′, and R′ in terms of P, Q, and R.

$$R^{s''z''} Q^{s''y''} P^{x''s''} p^{m-3} + \frac{s''(s''-1)}{2} a y''z'' p^{m-3} = R^{s'z'} Q^{s'y'}$$

$$P^{x's'} p^{m-3} + xsp^{m-3} + \frac{s'(s'-1)}{2} a y'z' p^{m-3}$$

which gives

$$s''z'' \equiv s'z' \pmod p \text{ and}$$

$$s''y'' \equiv s'y' \pmod p.$$

Elimination of s′ gives

$$s''[y'z'' - y''z'] \equiv 0 \pmod p.$$

Since the lowest value of s″ is to be p, then

$$y'z'' - y''z' \text{ not} \equiv 0 \pmod p \qquad (15)$$

This includes the conditions given above.

Substitute in (12), (13), and (14) the values of P′, Q′, and R′ in terms of P, Q, and R, and reduce

$$\begin{aligned} &R^{z} Q^{y+\beta xz'} P^{x} \left[1 + \alpha z' p^{m-3} + a\beta \frac{z'(z'-1)}{2} p^{m-3}\right] \\ &+ k\beta z' \frac{x(x-1)}{2} p^{m-3} + a(yz' - y'z) p^{m-3} + kxy' p^{m-3} \\ &= R^{z} Q^{y} P^{x} \left[1 + k' p^{m-3}\right] \qquad (16) \\ &R^{z} Q^{y+\beta xz''} P^{x} \left[1 + \alpha z'' p^{m-3} + a\beta \frac{z''(z''-1)}{2} p^{m-3}\right] \end{aligned}$$

$$+k\beta z'' \frac{x(x-1)}{2} p^{m-3} + a(yz'' - y''z) p^{m-3} + kxy'' p^{m-3}$$

$$= R^{z+\beta' z'} Q^{y+\beta' y'} P^{x[1+\alpha' p^{m-3}]} + \beta' x' p^{m-3}$$

$$+a\beta' y' z p^{m-3} + \frac{\beta'(\beta'-1)}{2} a y' z' p^{m-3} \qquad (17)$$

$$R^{z'} Q^{y'} P^{a[y'z'' - y''z']} p^{m-3} = R^{z'} Q^{y'} P^{[x'+a'x]} p^{m-3}. \qquad (18)$$

A comparison of the members of these equations gives six congruences between the original and the transformed constants and the nine unknown quantities.

I. $\alpha xz' + a\beta x \frac{z'(z'-1)}{2} + k\beta z' \frac{x(x-1)}{2} + a(yz' - y'z) + kxy' \equiv k'x \pmod{p}$

II. $\beta xz' \equiv 0 \pmod{p}$

III. $\beta' z' \equiv 0 \pmod{p}$

IV. $\beta' y' \equiv \beta xz'' \pmod{p}$

V. $\alpha xz'' + a\beta x \frac{z''(z''-1)}{2} + k\beta z'' \frac{x(x-1)}{2} + a(yz'' - y''z) + kxy'' \equiv \alpha' x + \beta' x' + a\beta' y' z + \frac{\beta'(\beta'-1)}{2} a y' z' \pmod{p}$

VI. $a(y'z'' - y''z') \equiv x' + a'x \pmod{p}$.

In case two of these groups are simply isomorphic, either one may be transformed into the other; the necessary and sufficient condition for this is that these congruences should be consistent and shall admit of solution for x, y, z; x', y', z'; and x'', y'', z''; where $x \not\equiv 0$ and $y'z'' - y''z' \not\equiv 0 \pmod{p}$.

In discussing these congruences the simplest cases are considered first, and we associate with them all cases into which they may be transformed.

The groups are divided into two sets, A and B. Each one of these sets is subdivided into 8 cases.

The results are given in tabular form below, and type forms are also given. In explanation of this double-entry table, $\equiv$ and $\not\equiv$ denote $\equiv 0$ and not $\equiv 0 \pmod{p}$.

	k	oc	a	($\beta \equiv 0$) A	($\beta \not\equiv 0$) B
1	$\equiv$	$\equiv$	$\equiv$		
2	$\not\equiv$	$\equiv$	$\equiv$		
3	$\equiv$	$\not\equiv$	$\equiv$	A2	
4	$\equiv$	$\equiv$	$\not\equiv$	A1	B3
5	$\not\equiv$	$\not\equiv$	$\equiv$	A2	B2
6	$\not\equiv$	$\equiv$	$\not\equiv$	A2	B2
7	$\equiv$	$\not\equiv$	$\not\equiv$	A2	*
8	$\not\equiv$	$\not\equiv$	$\not\equiv$	A2	B2

* B1 divides into two parts; the part where $\alpha - a\beta \equiv 0$ being simply isomorphic with B1, while the part where $\alpha - a\beta$ not $\equiv 0$ is simply isomorphic with B3.

The type groups are

A (1) k=p α=p a=p β=p the Abelian group of type (m—2, 1, 1)

A 2, k=1 α=p a=p β=p

B 1, k=p α=p a=p β=1

B 2, k=1 α=p a=p β=1

B 3, k=p α=1 a=p β=1.

The special forms the congruences take in these cases and from which these conclusions are drawn are given below.

A 1.

I. $k' \equiv 0 \pmod{p}$

III. $\beta' z' \equiv 0 \pmod{p}$

IV. $\beta' y' \equiv 0 \pmod{p}$

V. $a' \equiv 0 \pmod{p}$

VI. $x' + a' x \equiv 0 \pmod{p}$

III and IV give with condition (15) $\beta' \equiv 0$.

A 2.

I. $kxy' \equiv k' x \pmod{p}$

V. $ky'' \equiv a \pmod{p}$

VI. $x' + a' x \equiv 0 \pmod{p}$

B 1.

I. $k' \equiv 0 \pmod{p.}$

II. $\beta z' \equiv 0 \pmod{p}$ $z' \equiv 0$ from condition (15) y' and z'' not $\equiv 0$

III. $\beta' z' \equiv 0 \pmod{p}$

IV. $\beta' y' \equiv \beta x z'' \pmod{p}$ β' not $\equiv 0$

V. $a' x + \beta' x' \equiv 0 \pmod{p}$

VI. $x' + a' x \equiv 0 \pmod{p}$.

Elimination between V and VI gives

$$a' - a' \beta' \equiv 0 \pmod{p}$$

B 2.

I. $ky' \equiv k' \pmod{p}$

V. $k\beta z'' \frac{x(x-1)}{2} + kxy'' \equiv a' x + \beta' x' \pmod{p}$

VI. $a' x + x' \equiv 0 \pmod{p}$

B 3.

I. $k' \equiv 0 \pmod{p}$

V. $\alpha x z'' \equiv a' x + \beta' x' \pmod{p}$

VI. $x' + a' x \equiv 0 \pmod{p}$.

Elimination of x' between V and VI gives

$$a' - a'\beta' \equiv \alpha z''$$

and since z'' not $\equiv 0$

$$a' - a\beta'' \text{ not} \equiv 0 \pmod{p}$$

In the groups under A, $\{P\}$ is self-conjugate, under B, non-self-conjugate.

THE BIOGRAPHY OF VESPASIAN BY SUETONIUS NOTES FROM EPIGRAPHICAL SOURCES[1]

By Fred. B. R. Hellems

The notes are given in the order in which they would appear in an edition of the biography. The Suetonius references are to Roth's edition of 1898.

Vespasia Polla. Suet. Vesp. 1; p. 224, l. 16.

C. Caesari Augusto Ger*manico*, | Ger*manici Caesaris* f., Ti. Caes*aris Aug. n.*, | *divi Augusti pron.*, *pont.* max., trib. p*ot.* - - - -, *cos.* - - - - | *Vespasia* - - f. Polla.

(Arch. Epig. Mitth. aus Öster, XV (1892), p. 34.)

Although the inscription is very fragmentary there is little doubt that Bormann is right in assigning it to the mother of Vespasian. Its interest lies in the dedication to Caligula, under whom Vespasian was rather a favorite. Cf. Vesp. 2; p. 225, l. 24 seq. As to the particular object of the dedication we are left uninformed.

Flavius Sabinus. Suet. Vesp. 1; p. 224, l. 17.

- - - - - *leg. divi Claud*i pro pr. provin*c. Moe*|*siae*, *cur. census* Gallici, praef. urb*i* | *iterum. Huic* senatus, auctor*e* | *Imp. Caes. Vesp*asiano fratre, | *clupeum po*suit vadimon*is* | *honoris cau*sa dilatis, *fu*|*nus censorium* censuit, sta*tuam*|*in foro divi* Augusti *ponen*|*dam decrevit.* Dessau, 984.

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[1] To supplement the provokingly meagre literary evidence for the important reign of Vespasian I long ago began a collection of the epigraphical evidence. This task, however, I was compelled to resign to Mr. H. C. Newton, and the results of his research have been published in the Cornell Studies in Classical Philology, No. XVI. In the present paper I have tried to bring together such of the inscriptions as seemed directly pertinent in a commentary on the life in Suetonius. Naturally, I cannot separate Mr. Newton's work from my own; but I am sure that if the notes happen to be useful to any readers of Suetonius there will be no quarrel between us.

The restorations are by Mommsen. This is the only inscription that can be assigned indubitably to Vespasian's brother, who is often confused with T. Flavius Sabinus. Unfortunately it tells us little that is new.

Reate Vespasianus natus est in Sabinis ultra Reate vico modico, etc. Suet. Vesp. 2; p. 225, l. 10.

That Vespasian did not forget his native district is shown by the fact that the name of Reate appears in no less than five inscriptions connected with the settling of colonists by the emperor after the earlier wars of this reign. C. I. L. X, 4682; 4683; 4684; 4685; 4689.

Caenis. Post uxoris excessum Caenidem, Antoniae libertam et a manu, dilectam quondam sibi revocavit in contubernium habuitque etiam imperator paene iustae uxoris loco. Suet. Vesp. 3; p. 226, l. 2.

Dis manib. | Antoniae Aug. | l. Caenidis | optumae patron., | Aglaus l. cum Aglao | et Glene et Aglaide, | filiis. C. I. L. VI, 12037.

By the caprice of history not only this inscription to Vespasian's concubine has been preserved, but also an inscription to Flavia Helpis, her maid.

D. m. | Fl(aviae) Aug. lib. Helpidi | Caenidianae, quae | cum vixit dea et sanctissima | dicta est, Callistus Aug. lib. | Hyginianus coniugi caris|simae, cum qua vixit bene an|nis L, et Ulpia Calliste, filia, matri pi|entissimae, libertis, libertabus, pos|terisque eorum. H(oc) m(onumentum) h(eredem) n(on) s(equetur).

C. I. L. VI, 18358.

Licinius Mucianus. Suet. Vesp. 6; 13; p. 229, l. 5; p. 231, l. 29. We may assign his third consulship definitely to the year 72 A. D. C. I. L. VI 2016; 2053. The statement of Pliny (XII § 9; XIII § 88) that Mucianus was legate of Lycia is confirmed and the incumbency fixed under Nero by an inscription quoted in the Bull.

de Corr. Hellénique, 1886, p. 218. The appearance of his name on several pipes from Aricia makes it probable that he owned an estate in that favored neighborhood where Vitellius had dallied when he should have been preparing for his struggle for the empire.

C. I. L., XIV, 2173; Tac. His. III, 36.

Censorship. Suet. Vesp. 8; p. 229, l. 33. The vexed question of the exact date of his censorship can only be cleared up by assuming that the fragmentary C. I. L. V; 4312 read C*ensor des.* This would allow us to take April 1st, 73, as the day on which he assumed the censorship, and I think this date may be accepted as practically certain. It was during his censorship that he conferred Latin rights on Spain in connection with which there are many inscriptions.

Vide Newton op. C. p. 31 seq.

Deformis urbs veteribus incendiis ac ruinis erat. Suet. Vesp. 8; p. 230, l. 19. As we should have expected Vespasian proceeded most promptly to the practical work of restoring and improving the city. The streets and acqueducts received immediate attention, and before the end of 71 we have these two inscriptions.

Imp. Caesari | Vespasiano Aug., | pont. max., tr. pot. III, | imp. IIX, p. p., cos. III, des IIII, | s. c., | quod vias urbis | neglegentia | superior. tempor. | corruptas in | pensa sua restituit.

C. I. L. VI, 931.

Imp. Caesar Vespasianus August., pontif. max., trib. pot. II, imp. VI, cos. III, desig. IIII, p. p., | aquas Curtiam et Caeruleam, perductas a divo Claudio et postea intermissas dilapsasque | per annos novem, sua impensa urbi restituit. C. I. L. VI, 1257.

In connection with these words of Suetonius quoted above there is a peculiar significance attaching to C. I. L. VI, 940, for the *subrutores* were probably kept busy from the accession of Vespasian to the death of Titus.

Pro salute | T. Caesaris Aug. f. Imp. Vespasiani, | Ti. Claudius Clemens fecit. | T. Naevius Diadumen(us) cur(ator) col(legii) subrutor(um) | cultor(um) Silvani p(ecunia) s(ua) r(efecit?)
C. I. L. VI, 940.

There is no doubt that the *subrutores* were "wreckers," who cleared up the ruins of buildings that had suffered from fire or other causes.

Templum Pacis. Suet. Vesp. 9; p. 230, l. 28. We have no inscription connected with the Templum Pacis; but C. I. L. VI, 935, was engraved on the epistyle of the Templum Sacrae Urbis. C. I. L. 935 certainly does not belong to a temple of Saturn as stated by Schiller (Geschichte der röm. Kaiserzeit, p. 517, n. 6); nor do I believe that it is at present safe to assign C. I. L. VI, 937, to the temple of Saturn, although it is so referred even by Kiepert and Hülsen (Formae Urbis Romae, p. 87).

Vespasian's work in restoring various shrines won him the designation "restitutor aedium sacrarum." C. I. L. VI, 934.

Senatorial and equestrian orders. Amplissimos ordines, et exhaustos caedi varia et contaminatos veteri negligentia, purgavit supplevitque recenso senatu et equite, &c. - - - - Suet. Vesp. 9; p. 230, l. 33 seq. Many inscriptions show his work in this connection, and one should note the following among the citizens *adlecti inter praetorios*. Cornutus Tertullus, so well known from the letters of Pliny, C. I. L. XIV, 2925; L. Baebius Avitus, C. I. L. VI, 1359; Nonius Bassus (cf. next. note) C. I. L. IX, 5533; Plotius? Firmus. C. I. L. XI, 18321. I am almost sure that this is the rather remarkable Plotius Firmus mentioned as a friend and supporter of Otho in Tac. Hist. I 46, 82; II 46, 49. He probably passed into the service of Vespasian after Otho's death, which he had tried so hard to prevent.

Salvius Liberalis Nonius Bassus. Suet. Vesp. 13; p. 231, l. 33 seq. An inscription of peculiar interest records the career of this free-

speaking, many-named advocate — C. *Salv*io C. f. Vel. Liberali— *Nonio* Basso, cos., procos. provin | *ciae Ma*cedoniae, legato Augustorum | *provinc*. Britann, legato leg. V Maced., | *fratr*i *A*rvali, *adlecto* ab divo Vespasiano | *et divo Tito* inter tribunicios, ab isdem | allecto inter praetorios, quinq. IIII, p. c. Hic sorte | *procos. fac*tus provinciae Asiae se excusavit. C. I. L. IX, 5533.

It has never been pointed out, to my knowledge, that his wife may have been related to Vitellius as is suggested by the following epitaph: Vitelliae | C. f. Rufillae | C. Salvi Liberalis cos. (uxori) | flamini Salutis Aug., matri | optumae | C. Salvius Vitellianus vivos. C. I. L. IX, 5534.

Mestrius Florus. Suet. Vesp. 22; p. 234, l. 15. Mestrius Florus of the "au" episode is mentioned in an Ephesian inscription as proconsul. Bull. de Corr. Hell. I, p. 289.

GREEK SOURCES OF SHELLEY'S ADONAIS[1]

By George Norlin

"We are all Greeks," says Shelley, "our laws, our literature, our religion, our arts, have their roots in Greece."[2]

The essential truth in this enthusiastic utterance of Shelley will ever make a knowledge of Greek civilization an important part of the equipment of the scholar. It is not the knowledge of Greek civilization alone that is in question; it is the knowledge and understanding of our own, which presupposes and implies the Greek, one might almost say, as the day implies the sun.[3]

It was a favorite dictum of Max Müller that in order to understand what men are we must know what they have been. When we cease to understand the sources from which many of the highest products of our life have come, we lose the key to knowledge of ourselves. We cannot intelligently ignore the influences that have entered, however unobtrusively and unconsciously, into the life of the present and made us what we are.

From this point of view it is evidently sheer stupidity to call the study of Greek, the study of a language, a literature, a past, that is dead. It is rather the study of a past that has never died, or if, in a sense, it be said to have perished, it was only that it might again have life and have it more abundantly.

Particularly in some of the higher products of our civilization,

(1) After this paper was in the hands of the printer, I learned that its results have been in part, at least, anticipated by Dr. Richard Akermann, Quellen Vorbilder Stoffe zu Shelley's Poetischen Werken, Leipzig, 1890.

(2) Preface to Hellas.

(3) In this connection one might compare with the seemingly extravagant statement of Shelley the words of a thinker of very different type, Sir Henry Sumner Maine: To one small people * * * * it was given to create the principle of Progress. That people was the Greek. Except the blind forces of Nature, nothing moves in this world which is not Greek in its origin.

if I may so call our literature or art, has the Greek influence been a living and potent force, and, indeed, bids fair to continue to be. Not only are the forms of our literature Greek in their origin, but not seldom has an English writer caught and handed on the spirit of his master and model.

If this be true, then we of the universities are guilty of a strange inconsistency. We say that a man is not a good Latin scholar if he be not at the same time a good Greek scholar; that a man cannot know his Vergil unless he know his Homer, or his Horace unless he be acquainted with the Greek lyric poets. But we do not say, at least the requirements for higher degrees in our universities do not say, that a man in order to be an English scholar should be at the same time a Greek scholar. Why does not the same argument apply? If a man cannot intelligently understand his Vergil without knowing his Homer, then how can he fully appreciate his Milton, his Tennyson, his Matthew Arnold, his Shelley, and not be able to feel that they also were taught by the Muses of Helicon and had drunk of Hippocrene?

No thoughtful student of both literatures would deny, I think, that the debt of English literature to Greek, direct or indirect, is incalculably great; not that we can in every case put our hand on Greek influences in our English writers and say that it is there or here, for they are often too subtle and intangible for that, but in many cases they are so definite and clear that he who runs may read.

It is, perhaps, in the realm of pastoral poetry that we may see more clearly than anywhere else the influence of Greek models upon English verse. If the spirit often changes, the form remains essentially the same, and through the stubborn persistence of this form, with all its curious devices, we can, directly and indirectly, trace the lineage of the modern pastoral back to its Greek father, Theocritus and his immediate children, Bion and Moschus.

If one would take, for instance, one type of the Greek pastoral eclogue, the dirge, sung by one shepherd in lamentation over the misfortunes and death of another, and consider it in connection with the innumerable dirges in modern pastoral song, he could not, if he

would, shut his eyes to the family likeness that proves them all akin. To be more explicit, I mean Theocritus' Lament over the Death of Daphnis, Bion's Dirge for the Death of Adonis, Moschus' Lament for the Death of Bion, Vergil's Lament for Gallus in the Tenth Eclogue, and, to take only the best known English representatives, Spenser's Eleventh Eclogue and his Astrophel, Milton's Lycidas, Pope's Fourth Pastoral, Ambrose Philips' Third Pastoral, Gay's Friday or the Dirge, Matthew Arnold's Thyrsis, and Shelley's Adonais.

To point out the elements common to all these does not lie within the purpose of this paper. I shall content myself with considering only one of these types, the Adonais of Shelley, and showing the Greek elements in it, particularly the literary devices which he borrows directly from Theocritus, Bion and Moschus.

A mere juxtaposition of the English verses and the Greek will show without discussion how much he takes from these sources.

The particular poems in question are those mentioned in the above list, the first Idyll of Theocritus—a song of lamentation for the Sicilian shepherd-hero Daphnis; Bion's Song of Lamentation for the Death of Adonis; Moschus' Dirge over the Death of Bion. I shall refer to these simply as Theocritus, Bion, Moschus.

I follow as most convenient the order of Shelley's poem.

Stanza I.

I weep for Adonais—he is dead!
O, weep, for Adonais! though our tears
Thaw not the frost that binds so dear a head!
And thou, sad Hour, selected from all years
To mourn our loss, rouse thy obscure compeers,
And teach them thine own sorrow; say: With me
Died Adonais; till the Future dares
Forget the Past, his fate and fame shall be
An echo and a light unto eternity.

cf. Bion 1–5.

Weep for Adonis. The fair Adonis hath perished.
Perished has fair Adonis, and the Loves bewail his fate.
No longer sleep in thy robes of purple, Cypris. Rouse
thyself to sorrow, and in garments of mourning beat
thy breast and tell it to all: The fair Adonis hath
perished.[1]

cf. Moschus, 1–2, 7, 17, 18.

Sing woe ye forest glades and Dorian waters,
and rivers also, lament the lovely Bion. * * * A
fair minstrel hath perished. * * * * * Say to the Muses,
the daughters of Oeagreus, say to all the Thracian nymphs:
the Dorian Orpheus hath perished.[2]

Stanza II.

Where wert thou, mighty Mother, when he lay,
When thy son lay, pierced by the shaft which flies
In darkness? Where was lorn Urania
When Adonais died?

cf. Theocritus, 66.

Where were ye, Nymphs, when Daphnis was wasting
in death—O, where were ye?[3]

I need hardly call attention to the fact that Shelley makes

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(1) Αἴαζ' ὦ τὸν Ἄδωνιν· ἀπώλετο καλὸς Ἄδωνις,
ὤλετο καλὸς Ἄδωνις· ἐπαιάζουσιν Ἔρωτες.
Μηκέτι πορφυρέοις ἐνὶ φάρεσι Κύπρι κάθευδε·
ἔγρεο δειλαία κυανόστολε καὶ πλατάγησον
στήθεα καὶ λέγε πᾶσιν ἀπώλετο καλὸς Ἄδωνις

(2) Αἴλινά μοι στοναχεῖτε νάπαι καὶ Δώριον ὕδωρ,
καὶ ποταμοὶ κλαίοιτε τὸν ἱμερόεντα Βίωνα.
* * * * * * καλὸς τέθνακε μελικτάς
* * * * * * * * * * * * *
εἴπατε δ' αὖ κούραις Οἰαγρίσιν εἴπατε πάσαις
Βιστονίαις Νύμφαισιν, ἀπώλετο Δώριος Ὀρφεύς.

(3) πεῖ ποκ' ἄρ' ἦσθ', ὅκα Δάφνις ἐτάκετο, πεῖ ποκα Νύμφαι;

Urania, the Mighty Mother, the chief mourner over the death of Keats even as in Theocritus the Nymphs are the foster-mothers of Daphnis, and in Bion Aphrodite is the one who chiefly mourns the death of Adonis.

Stanza III.

Wake, melancholy Mother, wake and weep!
* * * * * *
* * * * * *
For he is gone where all things fair descend.

cf. Bion, 3, 4.

No longer sleep in thy robes of purple, Cypris.
Wake to thy grief * * * and beat thy breast.[1]

cf. Bion, 54–55.

Where Cypris (Aphrodite) says to Persephone, goddess of the realm of the dead:

"Take thou my lord. For thou art stronger far than
I, and all things fair descend to thee."[2]

Stanzas IV, VI.

Most musical of mourners, weep again!
Lament anew, Urania! He died,
Who was the sire of an unmortal strain,
Blind, old, and lonely when his country's pride
The priest, the slave and the liberticide
Trampled and mocked with many a loathèd rite
Of lust and blood; he went, unterrified,
Into the gulf of death; but his clear sprite
Yet reigns o'er earth; the third among the sons of light.

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(1) Μηκέτι πορφυρέοις ἐνὶ φάρεσι Κύπρι, κάθευδε,
ἔγρεο δειλαία * * καὶ πλατάγησον
στήθεα.

(2) λάμβανε. Περσεφόνα, τὸν ἐμὸν πόσιν. ἐσσὶ γὰρ αὐτά
πολλὸν ἐμεῦ κρέσσων, τὸ δὲ πᾶν καλὸν ἐς σὲ καταρρεῖ.

But now thy youngest, dearest one has perished,
The nursling of thy widowhood, who grew,
Like a pale flower by some sad maiden cherished
And fed with true-love tears instead of dew;
Most musical of mourners, weep anew.

cf. Moschus, 71–76.

This, most musical of rivers, is thy second sorrow, this, Meles,[1] is thy grief renewed. Homer of old did perish, that sweet voice of Calliope, and thou, they say, didst bewail thy fair son with tearful flood and fill all the sea with thy sad voice of grief. And now again, thou art bereaved of another son, and art wasted with a new sorrow.[2]

Stanza IX.

Oh, weep for Adonais! The quick Dreams,
The passion-wingèd ministers of thought,
Who were his flocks, whom near the living streams
Of his young spirit he fed, and whom he taught
The love which was its music, wander not—
Wander no more from kindling brain to brain,
But droop there, whence they sprung; and mourn their lot
Round the cold heart, where after their sweet pain,
They ne'er will gather strength, nor find a home again.

This is a striking example of the persistence of a literary tradition which begins with Theocritus. In the Lament for Daphnis, the hero of the song is a shepherd, and at his death his flocks and

(1) A river near Smyrna, here assumed to be the birthplace of Homer.

(2) *τοῦτό τοι ὦ ποταμῶν λιγυρώτατε δεύτερον ἄλγος,*
τοῦτο Μέλη νέον ἄλγος ἀπώλετο πρᾶν τοι Ὅμηρος,
τᾶνο τὸ Καλλιόπας γλυκερὸν στόμα, καί σε λέγοντι
μύρασθαι καλὸν υἷα πολυκλαύτοισι ῥεέθροις,
πᾶσαν δὲ πλῆσαι φωνᾶς ἄλα· νῦν πάλιν ἄλλον
υἱέα δακρύεις, καινῷ δ'ἐπὶ πένθεϊ τάκῃ

the wild beasts of field and mountain miss him and mourn his loss. [1]

Moschus closely imitates Theocritus and frankly makes a shepherd of his poet hero.

20–25.

He, the beloved of his flock, no longer sings his lay; no longer beneath the lonely oaks sings he; but in the realm of Pluto he sings the song of Lethe.

Voiceless are the mountains and his herds of kine wander and bemoan their loss and will not of the pasture.[2]

From this time on it becomes a literary convention to represent the subject of the dirge as a shepherd, missed and lamented by his flocks.

Shelley adopts the convention and turns it by a bold metaphor into one of the finest passages of his poem. His *shepherd*, also, had his flocks, "the quick Dreams, * * * whom near the living streams of his young spirit he fed." These "wander not—wander no more from kindling brain to brain. But droop there, whence they sprung; and mourn their lot round the cold heart, where, after their sweet pain, they ne'er will gather strength, nor find a home again."

Stanzas X, XI, XII.

And one with trembling hand clasps his cold head,
And fans him with her moonlight wings, and cries,

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[1] 71–75

Τῆνον μὰν θῶες, τῆνον λύκοι ὠρύσαντο,
Τῆνον χ'ὠ'κ δρυμοῖο λέων ἔκλαυσε θανόντα.
* * * * * *
πολλαί οἱ πὰρ ποσσὶ βόες, πολλοὶ δέ τε ταῦροι,
πολλαὶ δ' αὖ δαμάλαι καὶ πόρτιες ὠδύραντο.

[2] Κεῖνος ὁ ταῖς ἀγέλαισιν ἐράσμιος οὐκέτι μέλπει,
οὐκέτ' ἐρημαίῃσιν ὑπὸ δρυσὶν ἥμενος ἄδει,
ἀλλὰ παρὰ Πλουτῆι μέλος Ληθαῖον ἀείδει.
ὤρεα δ'ἐστὶν ἄφωνα, καὶ αἱ βόες ποτὶ ταύροις
πλαζόμεναι γοάοντι καί οὐκ ἐθέλοντι νέμεσθαι.

"Our love, our hope, our sorrow, is not dead;
See, on the silken fringe of his faint eyes,
Like dew upon a sleeping flower, there lies
A tear some dream has loosened from his brain."
Lost Angel of a ruined Paradise!
She knew not 'twas her own; as with no stain
She faded, like a cloud which had outwept its rain.

One from a lucid urn of starry dew
Washed his light limbs, as if embalming them;
Another clipt her profuse locks and threw
The wreath upon him, like an anadem,
Which frozen tears instead of pearls begem;
Another in her wilful grief would break
Her bow and winged reeds, as if to stem
A greater loss with one which was more weak;
And dull the barbèd fire against his frozen cheek.

Another Splendour on his mouth alit,
That mouth whence it was wont to draw the breath
Which gave it strength to pierce the guarded wit,
And pass into the panting heart beneath
With lightning and with music; the damp death
Quenched its caress upon its icy lips;
And, as a dying meteor stains a wreath
Of moonlight vapour, which the cold night clips,
It flushed through his pale limbs, and passed to its eclipse.

These stanzas are evidently suggested and elaborated from Bion, 79–86.

Beautiful Adonis reposes in robes of purple and about him the Loves make moan and weep, clipping their locks in grief for Adonis. One treads upon his arrows, another upon his bow, and another brings his well feathered quiver; one loosens the sandal of Adonis;

others bring water in golden vessels; one laves his thigh,[1] and another from behind fans with his wings the dying Adonis.[2]

Stanza XIV.

All he had loved, and moulded into thought
From shape, and hue, and odour, and sweet sound,
Lamented Adonais. Morning sought
Her eastern watch tower, and her hair, unbound,
Wet with the tears which should adorn the ground,
Dimmed the aërial eyes which kindle day;
After the melancholy thunder moaned
Pale Ocean in unquiet slumber lay
And the wild winds flew round, sobbing in their dismay.

This is an instance of another device that has become conventional in the dirge. Even inanimate nature is made to feel the spell of sorrow and join in the lamentation.

cf. Bion, 32–34.

All the hills sing woe for Adonis and the rivers bewail the sorrows of Aphrodite. The springs in the mountains also weep for Adonis.[3]

cf. Moschus, 28–29, and Theocritus, VII, 74–75.

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(1) i. e., the death wound inflicted by the tusk of the wild boar.

(2) *κέκλιται ἁβρὸς Ἄδωνις ἐν εἵμασι πορφυρέοισιν·*
ἀμφί δε μιν κλαίοντες ἀναστενάχουσιν Ἔρωτες,
κειράμενοι χαίτας ἐπ' Ἀδώνιδι· χὠ μὲν ὀϊστώς,
ὃς δ' ἐπὶ τόξον ἔβαιν', ὃς δ' εὔπτερον ἆγε φαρέτραν.
χὠ μὲν ἔλυσε πέδιλον Ἀδώνιδος· οἱ δὲ λέβησι
χρυσείοις φορέοισιν ὕδωρ· ὁ δὲ μηρία λούει·
ὃς δ' ὄπιθεν πτερύγεσσι ἀναψύχει τὸν Ἄδωνιν.

(3) *ὤρεα πάντα λέγοντι, καὶ αἱ δρύες αἰαῖ Ἄδωνιν·*
καὶ ποταμοὶ κλαίουσι τὰ πένθεα τᾶς Ἀφροδίτας,
καὶ παγαὶ τὸν Ἄδωνιν ἐν ὤρεσι δακρύοντι

Stanza XV.

Lost Echo sits amid the voiceless mountains,
And feeds her grief with his remembered lay,
And will no more reply to winds or fountains
Or amorous birds perched on the young green spray,
Or herdsmen's horn, or bell at closing day;
Since she can mimic not his lips, more dear
Than those for whose disdain she pined away
Into a shadow of all sounds; a drear
Murmur between their songs, is all the woodmen hear.

cf. Moschus, 30–31.

Echo grieves amid the rocks that thou art silent, and no longer does she mimic thy lips.[1]

Stanza XVI.

Grief made the young Spring wild, and she threw down
Her kindling buds, as if she Autumn were,
Or they, dead leaves; since her delight is flown,
For whom should she have waked the sullen year?
* * * * * * *

cf. Moschus, 31, 32.

In sorrow for thy death the trees cast down their fruit and all the flowers did fade.[2]

Stanza XVII.

Thy spirit's sister, the lorn nightingale,
Mourns not her mate with such melodious pain;
Not so the eagle, who like thee could scale
Heaven, and could nourish in the sun's domain

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(1) Ἀχὼ δ' ἐν πέτρῃσιν ὀδύρεται ὅττι σιωπῇς
κὐκέτι μιμεῖται τὰ σὰ χείλεα.

(2) σῷ δ' ἐπ' ὀλέθρῳ
δένδρεα καρπὸν ἔριψε, τὰ δ' ἄνθεα πάντ' ἐμαράνθη.

Her mighty youth with morning, doth complain,
Soaring and screaming round her empty nest,
As Albion wails for thee : * * *
* * * * * *

cf. Moschus, 37–45.

Not so much did the dolphin mourn by the shores of the sea, nor ever so sang the nightingale amid the cliffs; not so much mourned the swallow along the far stretching ranges of the hills, nor with such sorrow did the halcyon cry nor the sea bird sing, amid the grey-green waves; not so much did the bird of Memnon in the dells of dawn bewail the son of Morning, fluttering about his tomb, as they lamented the death of Bion.[1]

Stanzas XVIII–XXI.

Ah, woe is me! Winter is come and gone,
But grief returns with the revolving year;
The airs and streams renew their joyous tone;
The ants, the bees, the swallows, reappear;
Fresh leaves and flowers deck the dead Season's bier;
The amorous birds now pair in every brake,
And build their mossy homes in field and brere;
And the green lizard and the golden snake,
Like unimprisoned flames, out of their trance awake.

Through wood and stream and field and hill and Ocean,
A quickening life from the Earth's heart has burst,
As it has ever done, with change and motion,
From the great morning of the world when first

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[1] *οὐ τόσον εἰναλίαισι παρ' ἀόσι μύρατο δελφίν,*
οὐδὲ τόσον ποκ' ἄεισεν ἐνὶ σκοπέλοισιν ἀηδών,
οὐδὲ τόσον θρήνησεν ἀν' ὤρεα μακρὰ χελιδών,
ἀλικυονὶς δ' οὐ τόσσον ἐπ' ἄλγεσιν ἴαχε κήϋξ,
οὐδὲ τόσον γλαυκοῖς ἐνὶ κύμασι κηρύλος ἆδεν,
οὐ τόσον ἀῴοισιν ἐν ἄγκεσι παῖδα τὸν Ἀοῦς,
ἱπτάμενος περὶ σᾶμα, κινύρατο Μέμνονος ὄρνις.
ὅσσον ἀποφθιμένοιο κατωδύραντο Βίωνος.

God dawned on Chaos; in its stream immersed,
The lamps of Heaven flash with a softer light;
All baser things pant with life's sacred thirst,
Diffuse themselves and spend in love's delight
The beauty and the joy of their renewèd might.

The leprous corpse touched by this spirit tender,
Exhales itself in flowers of gentle breath;
Like incarnations of the stars, when splendour
Is changed to fragrance, they illumine death,
And mock the merry worm that wakes beneath.
Naught we know dies. Shall that alone which knows
Be as a sword consumed before the sheath
By sightless lightning? th' intense atom glows
A moment, then is quenched in a most cold repose.

Alas! that all we loved of him should be,
But for our grief, as if it had not been,
And grief itself be mortal! Woe is me!
Whence are we, and why are we? of what scene
The actors or spectators? Great and mean
Meet massed in death, who lends what life must borrow.
* * * * * *

This is an elaboration of the idea briefly and simply expressed in the finest lines of Moschus, 106-111.

Ah me! the mallows when they fade and perish in the garden, and the green parsley and the fair flowering tendrils of the anise, they awake to life again and grow, with the coming of another spring.

But we, the human kind, the great, the mighty, and the wise, when once we die, unheeding in the hollow earth we sleep—the long, endless, never waking sleep.[1]

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(1) Αἰαῖ ταὶ μαλάχαι μὲν ἐπὰν κατὰ κᾶπον ὄλωνται,
ἠδὲ τὰ χλωρὰ σέλινα τό τ' εὐθαλὲς οὖλον ἄνηθον
ὕστερον αὖ ζώοντι καὶ εἰς ἔτος ἄλλο φύοντι·
ἄμμες δ' οἱ μεγάλοι καὶ καρτεροί, οἱ σοφοὶ ἄνδρες,
ὁππότε πρᾶτα θάνωμες, ἀνάκοοι ἐν χθονὶ κοίλᾳ
εὕδομες εὖ μάλα μακρὸν ἀτέρμονα νήγρετον ὕπνον.

Stanzas XXII–XXIV.

He will awake no more, oh, never more !
"Wake thou," cried Misery, "childless Mother, rise
Out of thy sleep, and slake, in thy heart's core,
A wound more fierce than were his tears and sighs."
And all the Dreams that watched Urania's eyes,
And all the Echoes whom their sister's song
Had held in holy silence cried, "Arise ! "
Swift as a thought by the snake memory stung,
From her ambrosial nest the fading Splendour sprung.

She rose like an autumnal night, that springs
Out of the East, and follows wild and drear
The golden Day, which, on eternal wings,
Even as a ghost abandoning a bier,
Has left the Earth a corpse;—sorrow and fear
So struck, so roused, so rapt, Urania,
So saddened round her like an atmosphere
Of stormy mist; so swept her on her way
Even to the mournful place where Adonais lay.

Out of her secret Paradise she sped,
Through camps and cities rough with stone, and steel,
And human hearts, which to her airy tread
Yielding not, wounded the invisible
Palms of her tender feet where'er they fell;
And barbèd tongues, and thoughts more sharp than they
Rent the soft form they never could repel,
Whose sacred blood, like the young tears of May,
Paved with eternal flowers that undeserving way.

This is an imitation of the lines of Bion, which describe the distraction of Aphrodite when she awakens to the death of Adonis.

3-5. No longer sleep in thy robes of purple Cypris. Arouse

thee from sleep to grief and beat thy breast and tell it to all. The fair Adonis hath perished.[1]

16, 17. A cruel wound hath Adonis in his thigh, but greater is the wound in Aphrodite's heart.[2]

19, 24. Aphrodite unbinding her braided tresses, goes wandering through the forest, in her distracted grief, not tasting food, with feet unsandaled; and as she wanders on, the thorns pierce her feet and, "pluck the blossoms of her sacred blood." With shrill moans of grief, she hastens on through the long vales, calling again and again for her Assyrian lord.[3]

35. "And the flowers flush red with anguish,"[4]

64-66. As many as are the drops of blood that flow from the wound of Adonis, so many are the tears of Aphrodite. And from the drops of blood springs up the rose, and from the tears, the wind flower.[5]

One will note how daintily and beautifully the English poet adapts the literal description of his Greek original to suit the purpose of his theme. The wild wood through which Aphrodite wanders

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(1) Μηκέτι πορφυρέοις ἐνὶ φάρεσι Κύπρι κάθευδε·
ἔγρεο δειλαία * * καὶ πλατάγησον
στήθεα καὶ λέγε πᾶσιν ἀπώλετο καλὸς Ἄδωνις.

(2) ἄγριον ἄγριον ἕλκος ἔχει κατὰ μηρὸν Ἄδωνις·
μεῖζον δ' ἁ Κυθέρη φέρει ποτικάρδιον ἕλκος.

(3) ἁ δ' Ἀφροδίτα
λυσαμένα πλοκαμῖδας ἀνὰ δρυμὼς ἀλάληται
πενθαλέα νήπαστος ἀσάνδαλος, αἱ δὲ βάτοι νιν
ἐρχομέναν κείρουσι καὶ ἱερον αἷμα δρέπονται·
ὀξὺ δὲ κωκύοισα δι' ἄγκεα μακρὰ φορεῖται,
Ἀσσύριον βοόωσα πόσιν καὶ πολλὰ καλεῦσα.

(4) ἄνθεα δ' ἐξ ὀδύνας ἐρυθραίνεται·

(5) δάκρυα δ' ἁ Παφίη τόσσ' ἐκχέει, ὅσσον Ἄδωνις
αἷμα χέει· τὰ δὲ πάντα ποτὶ χθονὶ γίνεται ἄνθη
αἷμα ῥόδον τίκτει, τὰ δὲ δάκρυα τὰν ἀνεμώναν

distraught, the brambles that pierce her tender feet, become for Urania, who reflects the delicate and sensitive nature of Keats.

"Camps and cities rough with stone and steel,
And human hearts which to her airy tread,
Yielding not, wounded the invisible
Palms of her tender feet where'er they fell;
And barbed tongues and thoughts more sharp than they," etc.

Stanzas XXV, XXVI.

* * * * * *

"Leave me not wild and drear and comfortless
As silent lightning leaves the starless night!
Leave me not! cried Urania;

* * * * * *

"Stay yet awhile! speak to me once again;
Kiss me, so long but as a kiss may live;
And in my heartless breast and burning brain
That word, that kiss shall all thoughts else survive,
With food of saddest memory kept alive.
Now thou art dead, as if it were a part
Of thee, my Adonais! I would give
All that I am to be as thou now art,
But I am chained to Time, and cannot thence depart!

cf. Bion, 40-53. When she saw, when she marked the irremediable wound of Adonis, when she saw the dark blood upon the fainting thigh, she lifted up her arms and cried in sorrow: "Abide with me, Adonis; Abide, ill-starred Adonis, that I may come to thee for the last time, that I may cast my arms about thee and place my lips to thine.

Wake only for a little, Adonis, and kiss me one last kiss, kiss me only so long as a kiss may live. * * * And I will drink thy love, and this kiss I shall cherish, even as I cherish thee, Adonis, now that thou, ill-fated, dost flee from me, far from me dost thou flee, Adonis; down to Acheron thou goest, to the loathed and cruel

King of Death. But I, ah woe is me! must live, for I a goddess am, and may not follow thee.[1]

Stanza XXVII.

"O gentle child, beautiful as thou wert,
Why didst thou leave the trodden paths of men
Too soon, and with weak hands though mighty heart
Dare the unpastured dragon in his den?
* * * * * *

This is an evident reference to Aphrodite's reproach of Adonis for his bold fondness for the chase in which he met his death by the tusk of the wild boar.

Bion, 60, 61. Why wert thou overbold to follow the hunt: Why wert thou, who art so fair, mad to fight with the wild beast?[2]

The wild beast of Shelley, "the dragon in his den," evidently refers to the reviewers, who by their harsh criticism of Keats, hastened, according to Shelley, his death.

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(1) ὡς ἴδεν, ὡς ἐνόησεν Ἀδώνιδος ἄσχετον ἕλκος,
ὡς ἴδε φοίνιον αἷμα μαραινομένῳ περὶ μηρῷ,
πάχεας ἀμπετάσασα κινύρετο· μεῖνον Ἄδωνι,
δύτποτμε μεῖνον Ἄδωνι, πανύστατον ὥς σε κιχείω,
ὥς σε περιπτύξω καὶ χείλεα χείλεσ ιμίξω.
ἔγρεο τυτθὸν Ἄδωνι, το δ'αὖ πύματόν με φίλησον,
τοσσοῦτόν με φίλησον, ὅσον ζώει τὸ φίλημα,
* * * * * *
ἐκ δὲ πίω τὸν ἔρωτα, φίλημα δὲ τοῦτο φυλάξω
ὡς σ' αὐτὸν τὸν Ἄδωνιν, ἐπεὶ σὺ με δύσμορε φεύγεις,
φεύγεις μακρὸν Ἄδωνι, καὶ ἔρχεαι εἰς Ἀχέροντα
πὰρ' στυγνὸν βασιλῆα καὶ ἄγριον· ἁ δὲ τάλαινα
ζώω καὶ θεὸς ἐμμὶ καὶ οὐ δύναμαί σε διώκειν.

(2) τί γὰρ τολμαρὲ κυναγεῖς;
καλὸς ἐὼν τί τοσοῦτον ἐμήναο θηρὶ παλαίειν,

Stanza XXXVI.

Our Adonaïs has drunk poison—oh !
What deaf and viperous murderer could crown
Life's early cup with such a draft of woe ?
* * * * * *

This is an echo from Moschus' song, which contains the tradition that Bion met his death by poison given by some foe. Here again the literal poison of the Greek model becomes the poison of the reviewers' fatal pen.

116-119. Poison hath come, Bion, to thy lips. How could it touch thy lips and not be sweetened ? What man so cruel as to mix the fatal drug and give it to thy singing lips ? Verily he was an enemy of song.[1]

One could, perhaps, find other echoes in the Adonais from its Greek predecessors, but they are more vague and uncertain. Enough passages have already been cited to show how closely Shelley follows his models, and yet with what originality he adapts the old forms and conventions to the particular demands of his subject.

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[1] *Φάρμακον ἦλθε, Βίων, ποτὶ σὸν στόμα φαρμακοειδές.*
πῶς τευ τοῖς χείλεσσι ποτέδραμε κοὐκ ἐγλυκάνθη;
τίς δὲ βροτός, τοσσοῦτον ἀνάμερος ὡς κεράσαι τοί
ἢ δοῦναι λαλέοντι τὸ φάρμακον; ἢ φύγεν ᾠδάν.

A TRIPARTITE INTERVENTION IN HAYTI, 1851

By Frederic L. Paxson

It has generally been the policy of the United States to avoid entangling alliances with the Powers of Europe and to act independently of them in all matters of national interest. At times the United States has seen fit to act in co-operation with the European States, but in most cases this action has been independent, on parallel lines, rather than in combination with them. One of the rare instances in which the United States has deviated from this policy, occurred in 1851, when, in conjunction with Great Britian and France, an intervention was made in the affairs of Hayti and San Domingo. Without its futility and petty humiliations the intervention would have been sufficiently impolitic; but fortunately for the repute of the Fillmore administration it has gone unnoticed by the historians of the period.

With the various governments on the island of Santo Domingo the relations of the United States have always been intimate. From the time of the slave revolts and massacres of the period of the French Revolution, through the filibustering period, through the days of the Panama Congress and the anti-slavery petitions, through the era of annexation schemes, until the present day there has hardly been a five-year period in which some phase of its Dominican policy was not actively before the Department of State. Only too often the same Department has been unable to keep the questions from acquiring a popular standing until it became necessary to base their solution upon political considerations.

The island of Santo Domingo had been the key to the colonial empire in America which it had been Napolean's ambition to erect. And even after he had given evidence that he had relinquished the ambition he could not bring himself to surrender the island to its

inhabitants. Since the destruction of the army of Le Clerc by the blacks, the French control had been only in name, for the island had been in a state of anarchy with rival chieftains contending for supremacy and receiving aid and comfort from the filibustering expeditions that came openly from the United States. Unable to suppress either the blacks or these expeditions himself, Napoleon demanded that the United States government should do this for him, and three times within a year his minister Tallerand said "must" to Jefferson.[1] The President of the United States responded to the orders of the French Emperor, "to take the most prompt, as well as the most effective prohibitory measures"[2] to put a stop to these expeditions, and the administration's party pushed a bill through congress to check them.[3]

Revolution in the island continued to run its course in spite of France and the United States. Out of the chaos of petty chieftains arose one Dessalines, in 1804, to crown himself at Port-au-Prince as James the First. Two years later Christophe, his successor by the right of assassination, obtained a control of the Haytian end of the island which he soon had to share with a rival chieftain Petion. Dividing Hayti by an east and west line, Christophe, who assumed the title of Henry I, in 1811, ruled the north, while Petion controlled the south; and the division endured for nearly fifteen years. Then a full-blooded negro, Henri Boyer, succeeded Petion in the south, and marched north across the island to Cape Haytien. The subjects of Christophe welcomed him, the troops mutined in his favor, Christophe himself died by suicide or assassination, and Boyer proclaimed himself President of Hayti on October 22nd, 1820.[4]

The administration of President Boyer lasted for twenty-four years. Within two years of his accession he obtained control of the

(1) Henry Adams, *History of the United States*, III, 90.

(2) Turreau to Secretary of State, 14 October, 1805, *American State Papers, Foreign Relations*, II., 725.

(3) *Annals of Congress*, 9th Cong. 1st session. Page 21 and following.

(4) *Niles Register*, XIX, 202, 220; B. C. Clark: *A Plea for Hayti, with a glance at her Relations with France, England and the United States, for the last sixty years.* Boston, 2 ed, 1853, P. 4; J. Redpath: *A Guide to Hayti.* Boston, 1861, P. 19; S. Hazard, *Santo Domingo, Past and Present; with a glance at Hayti.* New York, 1873, P. 162.

whole island by reducing San Domingo at its eastern end. And for more than two decades the white population of the east had to tolerate the burden of a black government. Then it broke the bond.

Hayti had been independent for forty years when Boyer met with the usual fate of the Latin-American dictator, abdicated in the face of a revolution, and took the well-beaten path to Jamaica and obscurity. In spite of its traditional policy of early recognition, the United States had never recognized the independence of the republic, for its population was black and the temper of the southern states would not permit such a reward to a revolted slave population. Instead, the mere mention that Hayti was to be invited, seems to have raised the opposition that kept the United States from participation in the Panama Congress.[1] Petitions for the recognition of Hayti during the thirties raised a clamor in the House equal to that of the petitions for abolition in the United States.[2] But now when the fall of Boyer was marked by the secession of San Domingo from the republic, there was a partial revulsion of opinion.

American politicians divided on Haytian questions as they were already divided on slave questions in the United States. The population of San Domingo was predominantly white, and so could command the sympathetic consideration of southern leaders, while abolitionists in the north were distressed at once by the injury to the one republic that proved, to their minds, the capacity of the negro for self-government, and by the triumph of a theory of secession. The literature on Hayti falls into two classes as it voices the sentiments of those two schools of thought.

The deposition and flight of Boyer occured in March of 1843.[3] The secession of the eastern end of the island came a year later, for the white population was tired of negro dominance, and, as a Spanish Catholic community, resented the adoption by the new Haytian government of a constitution containing a clause in favor of religious

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(1) *Compilation of Reports of Committee on Foreign Relations, United States Senate, 1789-1901*, Wash., 1901, IV., 12; cf. Henry Wilson, *Rise and Fall of the Slave Power in America*, I., 116.

(2) J. Q. Adams, *Memoirs*, X., 66; H. S. Legaré, *Works*, I., 322.

(3) *Hazard*, 170; *Redpath*, 20; Britannicus, *The Dominican Republic and the Emperor Soulouque*, Phila., 1852, P. 11.

freedom.[1] When the independence of San Domingo was declared on 27th February, 1844, the leaders of that republic had already applied to the United States for countenance and recognition. Neither of these was in the mind of Abel P. Upshur, then Secretary of State.[2] But when the explosion of the gun "Peacemaker" on the "Princeton" forced a change in Tyler's cabinet, the new Secretary, Calhoun, both a southerner and a secessionist, was favorably disposed towards the new government.

But the reports of the agent, Hogan, whom Calhoun sent to investigate the condition of the island brought no action by the Tyler administration. Perhaps the question of Texas was too absorbing to admit of attention being given to Hayti. At any rate the black republic and its white rival were left to work out their own revolutions for six years until Soulouque and Santana emerged as their respective leaders.

The Emperor Faustin, who had been Solouque until the elevation of his title in 1849, rose from slavery to the dictatorship of Hayti. Without political knowledge or experience he proved to have a will and considerable of administrative capacity, even if the words of his admirer must be discounted, that "his strength is at home; it is not too much to say, that there is not a town, village or hamlet in Hayti, however distant from the capital, that does not spontaneously and joyfully claim to honor him as chief, and to love him as a man."[3] Once settled in the government of Hayti, it became his ambition to unite the island once more under a single administration.

It is in connection with the designs of Faustin upon the independence of San Domingo that England, France and the United States became involved in the politics of the island. The excellent harbor of Samana Bay in the territory of San Domingo became an objective point in their diplomacy, which for several years around 1850 was interested in the control of the Caribbean and the Isthmus.

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(1) *Hazard*, 247: *Britannicus*, 31, 63; B. C. Clark, *Remarks upon United States Intervention in Hayti*, Boston, 1853, P. 18-23.
(2) Clark, *Plea.*, 36.
(3) Clark, *Plea.*, 46.

Begging a protectorate and offering a lease or even a cession of Samana Bay in return for it, San Domingo was able to arouse at once the cupidity of the European Powers and the jealousy of the United States.

Pedro Santana became Dictator of the eastern republic in 1849, having risen to that position through the incapacity of his predecessor Jimines. His diplomacy aimed at recognition and a European protectorate, to which end an agent had already been sent to Europe. Baez, the agent, was not received in Spain at all; in France he negotiated a treaty that the Chambers would not allow even to be read; while Great Britian decided to deal directly with his government through a consul-general whom she commissioned to San Domingo in August, 1848.[1] The presence of a Haytian army within the territory of San Domingo in the spring of 1849 inspired Santana, and Baez, who had returned home to become President of Congress, to new efforts for a protectorate. But neither Schomburgk, the British consul-general, nor Rayband, the French consul-general at Hayti, was ready to take the necessary action, while the arrival of an American agent on the scene lent color to the belief that the United States was prepared to take a hand in the matter.

Benjamin F. Green, the son of Duff Green, and a southern man, was sent to San Domingo by Clayton in the administration of President Taylor, with power to conclude a treaty of amity and commerce with that republic.[2] The condition of the treaty was that he should find existing there a government of assured stability; meaning a government not under the protection of any foreign power. Fearing that the knowledge of his mission might arouse false hopes, he travelled in a private capacity for a time, and took into his confidence only the leading politicians.

The conditions that Green found prevailing in the island in the summer of 1849 were not such as his instructions contemplated. Santana, victor over the Haytiens and Dictator was striving for

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(1) *Brittanicus*, 24.

(2) The correspondence relative to his mission is found in *Senate Executive Documents, No. 12, Thirty-third Congress, First Session.* Referred to hereafter as *Green*.

foreign protection. Baez, his rival, was favoring an annexation to France.[1] To prevent this latter action and to save the republic became the policy of the American agent. And the policy was successful even after Baez had forced Santana out of the government and taken his place.

As a southern man, Green sympathized with San Domingo in the struggle. The real question, he wrote home on 24th October, is whether black or white shall rule.[2] And he thought that if a recognition by the United States would help the feeble white republic, that that recognition should be promptly extended. The administration, while it felt that no maritime state should command the Isthmus,[3] felt also as Green did, and was ready to take reasonable steps to check the power of Faustin. It had opposed a foreign protectorate, but when, in February, 1850, San Domingo asked the three powers to intervene jointly in her behalf[4] Clayton did not long withhold his consent. Upon the return of Mr. Green, he replied, for the commission of the agent was about to expire, his correspondence and the name of a *chargé d àffaires* would be submitted to the Senate, and "if the nomination should be confirmed, the President would be prepared to co-operate with the governments of England and France."[5]

With the understanding that the United States would co-operate in the work, the British consul was instructed, late in the fall of 1850, to act with the French *chargé* and the United States *chargé* to San Domingo. The war was to be stopped. The Emperor was to be "menaced" by the agents with any punishment up to a blockade. Beyond this point the intervening powers were not yet ready to go.[6]

But the death of General Taylor in July, 1850, changed the complexion of the administration, and Daniel Webster the new, Secretary of State, was not so sure as Clayton had been that the time

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(1) *Britannicus*, 29.
(2) *Green*, 11.
(3) Taylor's 1st Annual Message, Richardson, *Messages*, V, 16.
(4) *Green*, 16, 19; Feb. 22, 1850.
(5) *Senate Executive Document, Vo. 113, Thirty-Second Congress, First Session.* Quoted as *Walsh*. Clayton to Bulwer, May 20, 1850.
(6) Instructions to T. N. Usher, December, 1850, *Walsh*, 8.

was ripe for recognition. No *chargé* was nominated by the President, but the policy of his predecessor was so far adhered to that Robert Walsh was sent to Hayti in January, 1851, to take his part in the joint intervention. Like the other agents he was authorized to persuade and threaten, and finally "conjointly with your colleagues, [to] require the Emperor to conclude a permanent peace with the Dominican government, upon the basis which you may jointly prescribe to him, or to consent to a truce with that government of not less than two years."[1]

Walsh arrived at Port au Prince, the seat of the Haytian government, on the 5th of February, 1851.[2] He found the intervention already started by a note of Raybaud and Usher[3] informing the Minister of Foreign Affairs that their respective governments had "agreed to adopt coercive measures" to prevent a new expedition into the territory of San Domingo. Within a week after the arrival of the American agent another note demanded of the "Duke of Tiberon," the Foreign Minister, a categorical answer to their proposition of a "*definitive treaty of peace, or a truce of ten years, between the empire of Hayti and the Dominican republic.*"[4]

The diplomacy of the Haytian government proved itself able to bear the burden of the displeasure of the Powers. When the agents called for an answer to their note they were told that constitutional limitations forbade compliance with their request; then that a committee of four would be appointed to come to an understanding with them.[5] We shall be glad to meet your committee, replied the agents, but our terms cannot be discussed, you have simply to accept or reject.[6] The matter is too serious for off-hand decisions, responded the minister, we must know your reasons.[7] We are surprised, answered the agents, at the repetition of your request, we cannot discuss our terms, but are willing to state our reasons.[8]

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(1) Webster to Walsh January 18, 1851. *Walsh*, 3, 4.
(2) *Walsh*, 9.
(3) December 19, 1850, *Walsh*, 15.
(4) Feb. 11, 1851, *Walsh*, 15.
(5) Feb. 21, 1851, *Walsh*, 20.
(6) Agents to Dufrere, Feb. 21, 1851, *Walsh*, 20.
(7) Dufrere to Agents, Feb. 24, 1851, *Walsh*, 21.
(8) Agents to Dufrere, Feb. 27, 1851, *Walsh*, 21.

But the joint debate that was held on the sixth of March, came to nothing, and a week later the Haytian Minister gave his definitive answer to the original demands, stating that His Majesty would summon the Haytian Chambers to submit the proposition to them, and would have the agents notified of their decision as soon as possible.[1]

This was far from the categorial answer that the agents had demanded, but they were unable to do anything but fume over it. Faustin had evidently seen that the Powers did not intend to make use of force against him, and had realized that he could play for time with impunity. The public threats failed to move him. The private scoldings of Walsh, and his intimations that filibusters from the United States might follow a refusal, had the same result. The proposition of the Powers was referred in due form to the Haytian Chambers when they convened on the 27th of March.[2]

In the chambers went on for three weeks what Walsh called the "solemn farce" of debating what the Emperor had already decided. Early in April came a rumor that Faustin would send his own envoys abroad and transfer the negotiations to foreign soil. Finally on the 19th came the definite answer rejecting both of the demands of the allies.[3]

The intervention was at an end. Walsh took passage on the French war steamer *Crocodile* and went to San Domingo, whence he shortly returned to the United States. The demands of the three greatest powers in the world had been rejected by an insignificant negro republic led by an illiterate negro military chieftain. The war was not resumed, but the disgrace of yielding to their dictation was avoided. As an extreme partisan of Hayti described it "vain menaces, vague threats, the blockade, and the solemn declarations of the three great Dictators, fell like snowflakes on the sea."[4]

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(1) Dufrere to Agents, Mar. 11, 1851, *Walsh*, 25.
(2) *Walsh*, 26.
(3) Walsh to Webster, April 23, 1851, *Walsh* 34.
(4) Clark, *Remarks*, 25.

THE EHRLICH DIAZO REACTION[1]

By James R. Arneill, A. B., M. D.,
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In a preliminary report of the commission appointed by Surgeon-General Sternberg to investigate typhoid fever during the late war, Vaughan states that the chief difficulty with which the commission had to contend was the utter worthlessness in numerous instances of the diagnoses of the volunteer physicians. As a result of inability and lack of opportunity to make blood examinations for plasmodia malaria and Widal's test for typhoid fever, thousands of cases of typhoid fever were diagnosed malaria, typho-malaria, continued fever, dengue, indigestion, and diarrhœa, and treated as such.

It may seem an extravagant statement, but it is none the less true, that if the diazo test had been applied in a routine way, and alone depended upon for the diagnosis of typhoid fever, the vast majority of these cases would have been correctly diagnosed.

Before recording my observations and deductions in detail, I will give a short résumé of the history and the chemistry of the diazo reaction.

In 1860, Peter Griess discovered and produced the azo dyes. He early made use of sulphanilic acid and naphthylamin salts for the demonstration of nitrous acid and its salts by means of color reactions. In 1875 the chemist Weselsky demonstrated phloroglucin by combining it with nitrodiazobenzol, the product being a red dye. Ehrlich believed that in the urine of certain diseases aromatic bodies were excreted which would become diazotized under proper chemical treatment. With this idea in mind, in 1882 he experimented upon the urines of a large number of infectious and non-infectious diseases, and obtained a specific diazo reaction, especially in the urines of ty-

[1] Reprinted by courtesy of the editor of The American Journal of the Medical Sciences, March, 1900.

phoid fever and tuberculosis. Unfortunately, in his earlier contributions Ehrlich gave very meagre and unsatisfactory directions regarding the preparation of his reagents, with the result that different investigators of ability experimented in a blind way with the test and arrived at erroneous conclusions. Penzoldt, Petri, and von Jaksch declared that it could be obtained in numberless diseases and conditions, even in normal urine; that all sorts of bodies, such as grape-sugar, acetone, bile-coloring matter, and a number of medicines would give the same reaction, and considered the test worthless as a prognostic and diagnostic measure. Among American authors Munson and Oertel deny its diagnostic worth, claiming that diacetic acid is the cause of the test, and that its diagnostic and clinical significance must be identical with the ferric chloride test. Instead of using a $\frac{1}{2}$ per cent. solution of sodium nitrite, as recommended by Ehrlich, these investigators used as high as a 5 per cent. solution. Greene found that when he used a concentrated solution of sodium nitrite a reaction similar to the diazo was obtained even in normal urine.

Mistakes were also made in the interpretation of the test. Edwards regarded the junction-ring—eosin to garnet—as the Ehrlich diazo reaction, and found it in so great a variety of diseases and conditions that he considered it useless as a diagnostic or prognostic sign. Edwards made the fatal mistake of disregarding the crucial part of the test, the color of the foam. Every investigator of experience is familiar with the fact that a red ring of varying tint is obtained in countless diseases. This, however, is no more the Ehrlich diazo test than a colored precipitate is Fehling's test.

A misunderstanding as to the exact color of ring and foam necessary for the production of a genuine Ehrlich diazo reaction has perhaps been the most prolific source of error. Burghart, for instance, claims that if tincture of opium, cascara sagrada, or hydrastis canadensis be added to urine a reaction similar to the diazo is obtained. I experimented with these drugs and found that the only one which suggested in color the diazo reaction at all was tincture of opium, which gave a salmon tint to the foam. When these drugs

are taken in medicinal doses and excreted in the urine the author says there is no possibility of mistaking the reaction for the Ehrlich diazo reaction. A large number of our patients have taken cascara for weeks at a time without giving the reaction. It has been claimed that the end-products of certain medicines excreted in the urine give the diazo reaction. It is true that such substances as aniline, naphthalin, phenacetin, lactophenin, and orthoform give a color reaction when treated with the diazo reagents. This has been called by a few authors the Ehrlich diazo reaction. These were diazo reactions, but not the Ehrlich diazo reaction. At the beginning of this article I wish to emphasize the distinction between diazo reactions and the Ehrlich diazo reaction. There are a large number of diazo reactions which are used to demonstrate certain known substances, such as phloroglucin, acetone, bilirubin, naphthalin, grape-sugar, peptone, and diacetic acid, while the Ehrlich diazo reaction is used to demonstrate an unknown substance. In the former the colors vary a great deal—purple, violet, blue, etc.—with a foam which is not characteristic. Grape-sugar gives a beautiful fuchsin pink color with an intensely red foam. The distinction, however, between the reaction and the Ehrlich reaction is that a fixed alkali is required in the grape-sugar test, as it does not occur with ammonia.

It is the experience of every trained observer in a long series of diazo tests with a great variety of diseases that countless varieties of colors and tints have been obtained in the rings and the foams. By the inexperienced many of these would have been called diazo tests. These colors are yellow, brown, orange, and salmon, and mixtures of the same. At times even the characteristic red ring is obtained, but on shaking the essential characteristic pink foam is absent. Urea, uric acid, kreatin, xanthin, sarcin, oxalic acid, hippuric acid, allantoin, and urine rich in urobilin and indican do not give the Ehrlich diazo reaction. Normal urine does not contain diazotizable substances. In 1884, Ehrlich's work was confirmed in a series of experiments by Lenhartz, of Leyden's clinic. Fisher, Brecht, Löwinson, Cnopf and Grundies, Escherich, See, Dohrendorff, Goldschmidt, Loewe, Georgiewski, Brewing, Roessingh, Piering, Rütimeyer, Simon, Ger-

hardt, von Noorden, all made important contributions confirming the value of this reaction both as a diagnostic and a prognostic agent in typhoid fever, and especially in tuberculosis. Fisher found the reaction constantly in measles and rarely in pneumonia. Brecht states that if the reaction appears in pneumonia it is to be regarded as of grave significance. Goldschmidt found the reaction constantly in miliary tuberculosis and typhoid fever. Brewing found this reaction of great value as a diagnostic and prognostic agent in puerperal fever and concealed septic processes, as well as in typhoid fever and tuberculosis. In surgical tuberculous affections, Pape found that after every operation the reaction which had been previously present disappeared, usually within three to five days. Warthin and Simon both emphasized the importance of the color of the foam in making the test, and found it of great value in the diagnosis of typhoid fever and the prognosis of tuberculosis. Kessel considers the reaction of great aid in diagnosing typhoid fever in children. Michaelis especially emphasized the value of the test in the prognosis of tuberculosis. He also found this reaction to be very constant in measles, and strikingly infrequent in German measles.

Notwithstanding the fact that a great mass of convincing evidence has been collected during the past seventeen years as to the value of the Ehrlich diazo reaction, especially as to the diagnosis of typhoid fever and the prognosis of tuberculosis, it is remarkable that the great majority of practising physicians in this country are not familiar with the reaction, and never make use of it in diagnosis or prognosis of disease.

The Ehrlich diazo reaction, like so many of our useful clinical tests—the tests for indican, bile, sugar, hydrochloric acid, lactic acid —is a color reaction, and depends upon the production of dyes by the chemical union of suitable organic substances with a diazo compound.

In carrying out this test the reagent is prepared according to the formulæ recommended by Ehrlich. We require two solutions, which are termed respectively solutions I. and II.

Reagent

Solution I.	Sulphanilic acid,	1
	Hydrochloric acid,	50
	Distilled water,	ad. 1000
Solution II.	Sodium nitrite,	0.5
	Distilled water,	ad. 100

To fifty parts of solution I. add one part of solution II. and shake. To a few c.c. of this mixture, and an equal quantity of urine, add a quantity of ammonia equal to about one-eighth of the combined volume of the mixed urine and solution, letting it run down the side of the test tube.

At the point of contact of the ammonia and the mixture, colored rings of various tints form, ranging from light yellow, through dark yellow, orange, and brown, to eosin or garnet, depending upon the urine. The formation of a red zone is an indispensable part of the true Ehrlich diazo reaction. It is also essential that, on shaking, the foam take on a pink color. This color varies considerably in its intensity, depending upon the strength of the reaction, from the palest rose to the deepest pink, but must not be any other color, such as salmon, orange, etc.

A third part of the reaction which Ehrlich considered important, consists in the separation of a greenish-black or violet-black precipitate, which forms a layer on the surface of the light-colored sediment when the tube has been allowed to stand for twenty-four hours. The changes which take place in this reaction may be represented graphically as follows:

(1) $NaNo_2 + Hcl = HNO_2 + NaCl.$

Sulphanilic acid. Diazobenzolsulphonic acid.

(2) $C_6H_4NH_2HSO_3 + HNO_2 = C_6H_5N_2HSO_4 + HO.$

The resulting fluid contains small quantities of diazobenzolsulphonic acid, to which the reaction is due, together with some excess of sulphanilic and hydrochloric acids which are indifferent or beneficial. The diazobenzolsulphonic acid unites with certain aromatic bodies in the urine to form analine dyes. The solutions should be kept in dark bottles. If the mixture of solutions I. and II. is not used

immediately it should be placed in a dark bottle and kept as cool as possible. In summer it can be used for one or two days, and in winter three to five days. The urine likewise should be as fresh as possible. The proportions of the mixtures have been varied by certain authors. Greene used 1 part of solution II. to 100 parts of solution I., and says that with this dilute mixture the orange and mixed reds and yellows mostly disappeared. According to his experience pulmonary tuberculosis and pneumonia fail to give the reaction with this high dilution, while they do with 1 to 40. The reaction still remained distinct and perfectly defined in septicæmia, typhoid fever, and advanced malignant disease.

Ehrlich has suggested a new method of performing the test, in which he mixes one volume of urine with from five to six volumes of absolute alcohol, filters, then adds the sulphanilic acid mixture to the filtrate. My series of tests have all been performed according to the first method, which is more convenient, less expensive, and sufficiently delicate.

The cause of this reaction is not known. It has been attributed to diacetic acid (Munson and Oertel) and to acetone (von Jaksch). Warthin and Spiethoff proved conclusively that diacetic acid does not give the diazo reaction where $\frac{1}{2}$ per cent. sodium nitrite solution is used.

In my series of cases there were fourteen of diabetes mellitus, eight of which gave good tests for acetone, and although numerous diazo tests were made according to Ehrlich's method the reaction was never found.

All authorities agree that this reaction is not dependent upon fever. In some of our cases of tuberculosis and carcinoma with subnormal temperature a strong reaction was found. However, most of the cases of tuberculosis have an intermittent fever when the reaction is present. In the cases of typhoid fever the reaction never continued after the decline of temperature to normal. That the reaction is dependent on substances excreted during the course of certain diseases, and not upon the height of the temperature, is well illustrated by most of the cases of croupous pneumonia in which

temperatures of 104° and 105° are very rarely accompanied by the diazo reaction; also by the fact that Gerhardt found the diazo reaction in half of his cases of afebrile typhoid. In nearly all cases there is evidence of a marked intoxication of some sort bordering on the typhoid state. There are some things which we know about this substance. It is soluble in water and alcohol, insoluble in ether, chloroform, benzol, xylol and carbon disulphide. Neither sugar of lead, silver nitrate, nor platinic chloride precipitate the substance from urine. Liquor plumbi subacetatis and milk of lime precipitate most of it. It exerts no reducing action on alkaline copper sulphate solution. In making the test it is best that the urine be as fresh as possible. However, urine which has been evaporated to a syrupy consistency will retain the capacity to give this reaction indefinitely (Michaelis). After nine months, when dissolved in an equal quantity of water, it gave the reaction very well (Clemens). In polyuria the reaction may disappear, but on concentration of the urine the test becomes positive. Various materials, such as bilirubin, urobilin, and carbol, interfere with the reaction somewhat, but can be removed by sugar of lead or animal charcoal. Burghart states that when preparations of tannic or gallic acid, or tincture of iodine (not the salts of iodine), are taken internally they prevent this reaction in patients who have previously shown for some time good diazo tests. This statement, if confirmed, is a very important one, as these preparations are often given in two of our most important diazo diseases—tuberculosis and typhoid fever. The author believes that these substances attack the diazo reagents and thus prevent the reaction. I am unable to refute or confirm these statements. Clemens declares that in the test-tube tannic acid precipitated or destroyed the substance, which was not recovered by the use of hydrochloric acid.

In this article I shall record the results of a critical analysis of the diazo tests made in the clinic of Dr. George Dock, University of Michigan, from 1893 to 1900. It is a sequel to the report of Dr. Warthin published in 1893. The method of performing the test has been the same as that used by him, and is practically the same as

that recommended by Ehrlich, except that forty parts of solution I. are used instead of fifty, and no attention has been paid to the green precipitates. The crucial point in our tests has always been the production of a pink foam after the characteristic red ring. During this period a diazo test, with objective records of the results has been made on practically all of the in-patients in the medical clinic, on some of them many times, and on a portion of the out-patients. The number of cases examined approaches eight hundred, while the number of tests made runs up into the thousands.

A large number of chronic and a limited number of acute diseases have been examined for this reaction. The method which I have followed in this investigation has been to note every positive diazo reaction found in the urine records during this period, and the dates of tests. The clinical records of these patients were then examined, the diagnosis of the disease found, and the severity and progress of the case considered. All other cases with similar diagnoses were investigated as to diazo. I also attempted to follow up the diazo tests in each case, in order to find out the duration of the reaction in the various diseases and also its prognostic worth and its value in indicating complications and relapses and its association with fever. Only in the cases of typhoid fever and some of the cases of tuberculosis was this done to any satisfaction, since in many out-patients only one test was made; in others only a few; while in others the tests were days apart.

In the accompanying table the diazo-reaction is practically limited to one acute and one chronic disease—typhoid fever and tuberculosis.

In all of the cases of typhoid fever the diazo tests were made almost daily from admission to the hospital till recovery. Unfortunately, as nearly always happens, it was impossible to follow the cases from the beginning, since some of them did not enter the hospital till the second or even the third week of the disease. Two of the three cases of typhoid fever in which diazo was negative entered the hospital in the second or third week of the disease, and had very light attacks. The Widal test was not made in either case. It is

Diseases.	Diazo present.	Diazo absent.	Total.
Abdominal tumor	2	5	7
Amyloid Disease	2	1	3
Bronchitis, chronic	1	10	11
Carcinoma	5	28	33
Diabetes	0	14	14
Erysipelas	0	1	1
Gastritis, chronic	1	157	158
German measles	0	2	2
Heart disease	2	22	24
Leukæmia	1	12	13
Malaria	1	1	2
Measles	0	2	2
Pneumothorax and empyema	2	6	8
Pneumonia (also tuberculosis)[1]	1	6	7
Rheumatism, acute (also heart disease)[1]	1	5	6
Rheumatism, chronic	1	9	10
Typhoid fever	19	3	22
Tuberculosis	42	40	82
	81	324	405

quite possible that the diazo was present and disappeared before entering the hospital. The third case gave a positive Widal reaction, but was an unusually mild attack. With the exception of the case just mentioned, all of our cases of undoubted typhoid fever that entered the hospital sufficiently early gave the diazo reaction.

In this series of cases the following sequels were met with: Phlebitis, two; abcess, three; pneumonia, one; hemorrhage from bowel, one. None of these sequels were heralded or accompanied by a diazo reaction. In two cases there were relapses with a reappearance of the diazo. The reappearance of the diazo points more to a relapse than the occurrence of a complication, according to the findings in our series of cases. Rütimeyer states that in relapses we almost always get a renewed reaction if the reaction has disappeared before the relapses occur. Michaelis affirms that the reappearance of the

[1] ("Also tuberculosis") refers only to the case in which diazo was present; the same is true of ("also of heart disease").

diazo reaction allows us to make a certain differential diagnosis between a recurrence and a secondary fever brought about by other causes.

The question often arizes, does the intensity or duration of the reaction correspond with the severity or length of the fever? We answer in the affirmative as to the duration of the reaction. If a patient with the clinical symptoms of typhoid fever comes under our observation toward the end of the second week of the disease and diazo is absent, the chances are that the infection is on the decline and we prognose a mild course. If, on the other hand, the diazo continues and increases in intensity, the case is liable to be prolonged and more serious. In our series of cases the diazo continued longest in the patients who were sick longest and whose symptoms were most pronounced. According to Ehrlich, the reaction first appears between the second and sixth days, and disappears usually during the first days of remission. In a case of typhoid fever which Dr. Dock was able to follow from the first day of the disease the diazo reaction was present for the first time on the fifth day. In one of our cases it was also present as early as the fifth day.

To illustrate the constancy with which this reaction is present in typhoid fever I will give the statistics of different investigators. In my series it was present in 19 out of 22 cases. Hewetson found it in 136 of 196 cases examined in Osler's clinic. The combined cases of Ehrlich, Spiethoff, Brecht, Brewing, Paterson, Jez, and Nissen number 178, of which 174 gave the reaction. Rivier has collected 536, of which 520 gave the reaction. Gerhardt says that in his clinic during a period of five years only one *bona fide* case of typhoid fever, which was proved by post-mortem, failed to give the diazo reaction. Zinn found the reaction in 75 per cent. of cases. Clemens found it in 135 of 157 cases. Greene obtained a positive result in 28 of 29 cases. Friedenwald found it in 20 of 21 cases. Dawson found it in 44 of 85 cases.

Out of a total of 82 of our cases of tuberculosis the diazo reaction was found in 42. The number of tests made in each case varied a great deal, inasmuch as a large number of the patients remained

in the hospital but a few days. Had they remained longer it is quite possible that in many cases reported as negative the reaction would have been found. In some of the positive cases there were intervals of considerable length during which diazo was absent. The test was almost never found in patients with slight lesions and slow course, but nearly always in those with advanced lesions and rapid course.

In several cases of the acute forms of the disease the presence of the reaction was responsible for a grave prognosis, notwithstanding the fact that the signs were very meagre and symptoms were practically absent. A striking illustration of this fact was the case of Mr. S., who came to the hospital with the diagnosis of dyspepsia. He had neither cough nor expectoration, but had a slight evening rise of temperature, with loss of weight and strength. On physical examination there were slight signs of infiltration of the right apex. The urine showed a good diazo. Dr. Dock, who had not examined the patient, but had seen him and noted his cachectic appearance, immediately predicted tuberculosis with a bad prognosis. Ammonium chloride and iodide of potassium were given to produce expectoration, and eleven days after admission tubercle bacilli were found in the sputum and several days later in the stools. The patient failed very rapidly, developed a severe diarrhœa, and remained in a typhoid state till death, which occured six weeks after admission to the hospital. Daily diazo tests were made from the day of admission till death, and, with the exception of five tests, were all positive, though varying somewhat in intensity.

In cases of pulmonary tuberculosis in which the diazo reaction is found continuously for some days, I believe, with Michaelis, that a grave prognosis should be made. This fact is of decided practical interest and should influence us in advising patients about change of climate. The amount of needless suffering which is caused, the waste of large sums of money uselessly by poor people, the harm done the climatic treatment of consumption by sending hopeless cases to our Western resorts to die is something criminal. It is by no means an easy matter to give advise in these cases. Many physicians are not thoroughly familiar with the physical signs of this disease

and know not what conclusions to draw from their examinations. All cases of tuberculosis giving the diazo reaction continuously for some days may be considered in the third stage of the disease. Such cases will not be permanently benefited by change of climate.

On an average between 20 and 30 per cent. of consumptives give the reaction (Clemens). The prognostic value of the reaction in this disease can be gathered from the fact that in 100 of Clemens' fatal cases 87 of them gave this test. Michaelis states that the great majority of consumptives who show a marked diazo for several days die within half a year. In 88 cases, which he had collected since 1896, 63 gave a positive, 25 a negative reaction. Of the 63 diazo cases, 50 died in the hospital, 5 left unimproved, 2 were transferred, and 6 improved. Of the 25 cases without diazo, 20 improved, 1 left cured, 2 died, and 2 did not improve.

The absence of the reaction in one of my cases of tuberculous meningitis and miliary tuburculosis should be emphasized because of the assistance it would render in making an early differential diagnosis between these diseases and typhoid fever, were it always absent in the former. Anyone with a large experience will admit that even a master medical mind stands in doubt in many cases in differentiating between typhoid fever so protean in character, with such variable temperature-curves, and tuburculous meningitis and miliary tuberculosis. In the above case it was impossible to make a diagnosis till the development of the brain symptoms. I made a Widal test, and with a dilution of 1 to 10 there was characteristic clumping, while with 1 to 30 the agglutination was not typical. Daily diazo tests were made, but the reaction was never found—a fact which had weight in eliminating typhoid fever. Post-mortem proved it to be a case of tuberculous meningitis and miliary tuberculosis. Simon says in his *Clinical Diagnosis:* "Since the reaction is obtained not later than the twenty-second day of the disease, and is usually present as early as the fifth or sixth day in typhoid fever, and while it generally does not appear earlier than the beginning of the third week, and then persists almost to the end in acute tuberculosis, its occurrence may be of decided value in diagnosis in many instances."

Three of the cases of tuberculosis in which the diazo was negative were cases of peritoneal tuberculosis. One of the cases in which it was negative was tuberculous meningitis and miliary tuberculosis. In one case of peritoneal tuberculosis the diazo was positive. In one case in which there were both pulmonary and peritoneal tuberculosis the diazo reaction was positive.

The differential diagnosis between cirrhosis of the liver and tuberculosis of the peritoneum is often difficult. The presence of this reaction favors tuberculosis, but does not assist in excluding cancer or sarcoma of the peritoneum.

In measles, statistics show the reaction to be very constant. Combining the statistics of Brewing, Brecht, Fisher, Nissen, Rivier, and Clemens we have ninety-eight cases in which there were only twelve negative. Michaelis uses this reaction as a means of differential diagnosis between measles and German measles, it being absent constantly in the latter. The reaction lasts about five days. In two of my cases of measles and several cases of German measles the reaction was absent.

Typhus Fever. Fischer found it in three cases; Dawson in five out of ten cases.

Scarlet Fever. Of eighty-seven cases from Berwing, Nissen, and Clemens thirty gave the reaction. In three typical cases of moderate severity I never found diazo.

In thirty-three of our cases of carcinoma it was present in five. Michaelis believes that in these cases it is due to a secondary infection, as, for instance, an ulcerating carcinoma and similar processes with bacterial infection.

Erysipelas. In a case with very extensive lesions originating at the site of drainage wounds and extending over almost the entire body, I was never able to find diazo. Clemens reports five positive cases in one hundred and twenty-two. Coste found the reaction in two-thirds of his cases.

The results of my observations have been to confirm in most particulars the statements of Ehrlich. They also agree with results published from this laboratory in 1893 by Dr. Warthin. The fact

that during the past eight years the test has been made in the same way in the clinical laboratory makes our results most trustworthy.

In deciding upon the clinical value of the Ehrlich diazo reaction we can fairly discard the work of all investigators who have not performed the test in accordance with the directions laid down by Ehrlich, or who have not considered the pink foam as the important factor in the test. In this list are included the names of Penzoldt, Petri, von Jaksch, Munson and Oertel, and Edwards. The statistics presented by numerous investigators who have performed the test correctly are overwhelmingly convincing as to the value of this reaction in the diagnosis and prognosis of typhoid fever, and the prognosis of diseases such as pneumonia, diphtheria, septicæmia, and especially tuberculosis.

THE OVERTURNS IN THE DENVER BASIN[1]

By Junius Henderson

Geological field work in the foothill region of the Denver Basin has possibly been hampered to some extent by the assumption that the overturn of certain formations—and the locally increased dip of the higher strata in other cases—were caused by the tangential, or nearly horizontal, pressure which is commonly supposed to have produced the mountain range. It is very possible that the direct effect of gravitation has not received sufficient consideration. Without in the slightest degree discrediting the lateral-compression theory of mountain uplift, of which there is other evidence along the foothills,

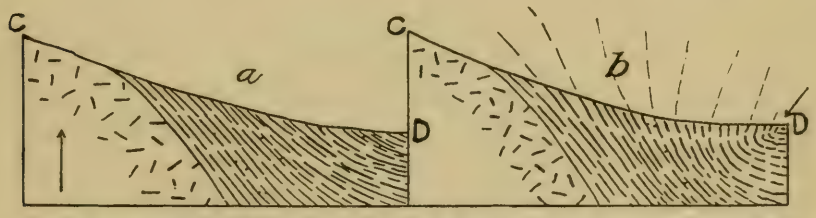

Fig. 1 (after figure in Monograph XXVII, U. S. Geol. Survey, p. 47).—*A* shows effect of vertical upward pressure, with dip of strata nearest the point of uplift greater than at a more distant point. *B* shows effect of tangential pressure.

it is worthy of notice that the overturning of strata flanking the foothills, may, at least in many places, and in every instance with which the writer is familiar, be, with good reason, ascribed to a very different cause.

In Monograph XXVII, United States Geological Survey, *Geology of the Denver Basin*, the fact is pointed out that the strata at some distance from the Archæan are generally tilted to a much higher angle than those lying nearer the granitic axis of the range, and this

(1) Reprinted by courtesy of the Journal of Geology, Vol. XI, No. 6, September-October, 1903.

is deemed an indication of tangential compression. Two diagrams are given to show the different effects of vertical upward and oblique downward pressure, which are here reproduced, Fig. 1.

Acceptance of that idea without further investigation led the writer and others at first to overlook certain phenomena, until the discovery of what appear to be Benton shales on the south side of Boulder Creek, disappearing under the apparently overturned Jura-Trias at the base of Flagstaff Mountain (the axis of the Boulder Arch, described in the monograph before mentioned), compelled a re-examination of the subject. In this vicinity the most pronounced overturn is in the Niobrara basal limestone, which is very hard and sufficiently resistant to form a ridge-making element. It is normally overlaid by several thousand feet of easily eroded Upper Niobrara and Pierre shales, and underlaid by Benton shales. When these formations are erected to a position approaching the vertical, the rapid cutting away of Upper Niobrara and Pierre shales must inevitably leave the Niobrara limestone partly unsupported on the east side, to bear the burden of the lateral pressure of the mountain column upon its base. Flagstaff Mountain rises abruptly about 1,000 feet above the upturned edge of the limestone. Other foothills are still higher, others still are lower and less abrupt, while beyond the foothills the main Rocky Mountain range towers to a height of from 10,000 to over 14,000 feet above the level of the sea.

There are reasons for supposing that at Flagstaff, as the unsupported limestone gave way and overturned, a break in the underlying Dakota (here very thin) and in Jura-Trias permitted the latter to swing outward at the base and inward and downward at the apex, thus executing a partial revolution on an axis. In the meantime, the yielding Benton shales crowded down into the opening thus made, and the broken edges of the Triassic, swinging outward, passed out over the Dakota, Benton, and Lower Niobrara in such a position as now to rest upon the overturned Niobrara shales, giving the impression at first glance that the Dakota, Benton, and Lower Niobrara had never been deposited, and that the Triassic had participated with the Upper Niobrara in the overturn. The following diagram, drawn by

Mr. H. F. Watts, of Boulder, Colo., who was associated with the writer in this work, will be an effectual aid to an understanding of what seems to have taken place.

After solving the problem at this point, it was easy to recognize the same phenomenon (which in local field parlance has been designated a "slump") at various points north and south for some distance. It frequently results in the production of a bench similar to the one on Flagstaff, locally known as Huggin's Park, but does not usually result in covering intervening formations on so extensive a scale.

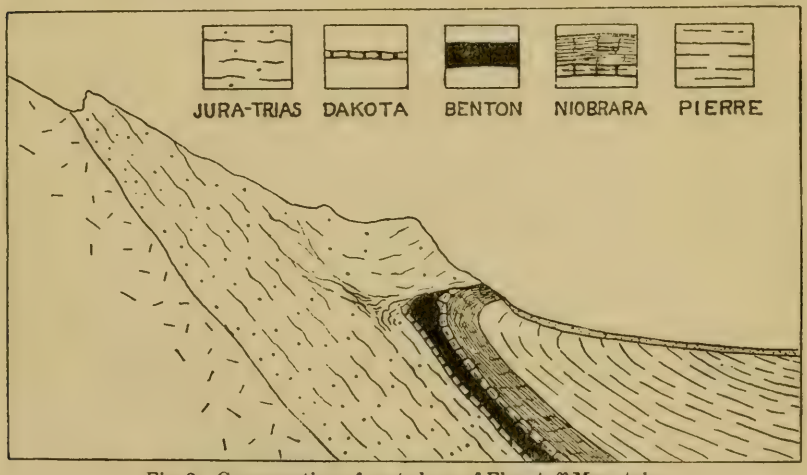

Fig. 2. Cross-section of east slope of Flagstaff Mountain.

Whether like conditions exist at all places in the Denver Basin where overturns occur, the writer is unable to say, not being familiar with the foothill region south of the Boulder county line; but the matter is worthy of further investigation before assuming that such overturns have any bearing upon the theories of mountain structure. The same process that caused the overturns in these cases, has presumably caused the greater dip of the later formations in cases where they have not been overturned.

Museum, University of Colorado,
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ON LAUGHTER[1]

By Arthur Allin

So far as ticklishness is concerned, a very important factor in the production of this feeling is undoubtedly that of the summation of stimuli. In a research of Stirling's carried on under Ludwig's direction[2] it was shown that reflex contractions only occur from repeated shocks to the nerve centers—that is, through summation of successive stimuli.

That this result is also due in some degree to an alternating increase in the sensibility of the various areas in question from altered supply of blood is reasonably certain. The connection of tickling with capillary pulsation is therefore worthy of investigation. As a consequence of this summation-process there could result in many cases and in cases of excessive nervous discharge the opposite of pleasure, namely, pain. This would result from long-continued stimulation or from light stimulation whenever the central nerve cells were possessed of little stability or inhibitory capacity, as in sickness, etc. A number of instances have been recorded of death resulting from tickling and there is no reason to doubt the truth of the statement that Simon de Montfort, during the persecution of the Albigenses, put some of them to death by tickling the soles of their feet with a feather. Mediæval justice and the hidden doings of the Inquisition might reveal many such instances if they were investigated. Lauder Brunton suggests that possibly the different effect of a slight stimulus like the touch of a feather, which causes intense reflex action, and of a gentle but steady pressure of the finger, which gives rise to no reflex action at all, may be due to the stimulation by the latter of two sets of nerves which counteract or inhibit each

(1) Reprinted from *The Psychological Review*, Vol. X, No. 3, May, 1903.

(2) Stirling, *Ludwig's Arbeiten*, 9ter Jahrgang, p. 290; *Sitz. Ber. d. k. Sach. Gesell. d. Wiss.*, Bd. XXVI., p. 439.

other.[1] It may be that the effect of steady pressure may cause a general diffused hyperæmia, whereas the stimulation resulting in the phenomena of tickling may and undoubtedly does cause a sudden convulsive hyperæmia which entails an explosive motor discharge. This relief of sudden congestion by additional stimulation of other and different nerve endings is observable in the relief afforded by rubbing or stroking a part which has been pinched or bruised, or by scratching an itching spot.

An additional causal factor in the production of tickling may lie in the nature and structure of the nervous process involved in perception in general. According to certain histological researches of recent years[2] we know that between the sense organs and the central nervous system there exist closely connected chains of conductors or neurons, along which an impression received by a single sensory cell on the periphery is propagated avalanche-like through an increasing number of neurons until the brain is reached. If on the periphery a single cell is excited, the avalanche-like process continues until finally hundreds or thousands of nerve cells in the cortex are aroused to considerable activity. Golgi, Ramon y Cajal, Koelliker, Held, Retzius and others have demonstrated the histological basis of this law for vision, hearing and smell, and we may safely assume from the phenomena of tickling that the sense of touch is not lacking in a similar arrangement. The importance of this law, it may be incidentally remarked, is manifest at a glance, for a future science of education. The spread of all methods whereby first-hand information is gained, while empirically found to be eminently satisfactory, is now known to rest upon a scientific basis. The laboratory method, kindergarten and primary object lessons, and constructive work, the use of illustrations in textbooks, magazines and newspapers, the stereopticon, etc., etc., may be cited as empirical recognition of this scientific fact.

May not a suggestion be offered with some plausibility, that even an ideal or representative tickling, where tickling results, say,

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(1) Lauder Brunton, 'On Inhibition,' 'West Riding Asylum Reports,' 1874, p. 179, and *Nature*, 1883, Vol. XXVII.

(2) Ramon y Cajal, 'Einige Hypothesen über den anatomischen Mechanismus der Ideenbildung, der Association und der Aufmerksamkeit,' *Archiv für Anatomie und Entwickelungsgeschichte*, Jahrgang 1895, pp. 367 ff.

from some one pointing a finger at the ticklish places, this avalanche-like process may be incited from central centers, thus producing, although in a modified degree, the pleasant phenomena in question? It would be in such a case another form of circular reaction.

Among the parts not mentioned by Sully[1] as subject to ticklishness might also be mentioned the palate and the lips or any part rendered more or less sensitive, as in the case of sores. The palate, in many cases at least, may be tickled by having the tip of the tongue pass lightly backwards or forwards over its surface. In certain physical moods such ticklishness with me is almost unbearable. The reactions observable upon the recovery by a limb of its normal condition after having been 'asleep' are identical in some respects with certain phenomena of tickling. A German child remarked in my hearing that champagne 'schmeckte ebenso wie eingeschlafene Füsse.'

In visual and auditory perception there may be induced some of the phenomena of tickling. A medical friend of mine informs me that certain notes in deep solemn music affect his epigastric muscles in a sort of shock reaction. The quivering can be induced by false notes at times. As to the deepest causal factor, I should say that tickling is the result of vaso-motor shock. In addition to these cases the phenomena of tickling may be autogenous in nature, that is to say, vaso-motor changes may be induced in the skin without apparent external stimulation. These changes are known at times to produce the phenomena of tickling.

If hypotheses are in order, I might suggest that as the attitude of disgust and dislike may be an incipient act of vomiting or the rejection of unpalatable food, so the smile may betoken an attitude of the whole organism in which the inception of food is the most striking characteristic. These actions which are obviously so useful in matters of food may have become in the course of social evolution associated with other affairs because of their eminently social symbolic value. The lower animals must perforce express themselves

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(1) Sully, Essay on Laughter, London, 1902.

somewhat differently because, according to the testimony of comparative anatomists[1] they lack the necessary facial muscles for language and the smile.

The laugh may have another physiological *raison d'etre* besides that mentioned by Darwin, Spencer and Sully, of relief of cerebral distension and congestion. Like singing, it may be a therapeutic agency in reference to pulmonary exercise, blood-oxygenation and general bodily nutrition. The deep inspirations which the singer and laugher are compelled to make cause a distension of a number of airvessels ordinarily in a condition of semi- or almost complete collapse. As a result of the laugh the circulation is hurried on through them and the lungs are developed to their fullest capacity. The well-developed lungs, by facilitating the process of oxygenation, favor the nutrition of the body in general. The laugh, it is true, causes 'a cessation of cerebal strain,' but the greatest relief is of pulmonary or vaso-motor origin. The sigh also possesses the same function, but the difference between the sigh and the laugh is the difference between work and play. As a general rule the play activities are more general and involve a greater amount of metabolism. The vitality of play is more intense. As singing has been recommended as a valuable adjunct in the treatment of anæmia and pithinoid chests, so laughter must not be denied its therapeutic and metabolic virtues. Deep inspirations favor the flow of blood through the lungs, from the right to the left side of the heart. Thus occasional sighs or laughs, or in other words deep inspirations, interrupt the shallow breathing constituting so-called 'breathless attention.' The shallow breathing leads to stagnation of blood in the right heart, and an occasional deep inspiration is necessary to relieve this. By holding the breath for a moment the stagnation of the blood in the right heart will provoke epigastric pulsation and cause the veins in the head and neck to swell. In cases of death from suffication or drowning, the right heart is found engorged with blood. Now in most instances of witticism or in joking, although not in all, there is an element of expectation, suspense or inhibited function. The laugh

(1) Cited by A. H. Keane, 'Ethnology.'

is the rehabilitation of function, the rebound to increased metabolism. This may also explain the easily-excited laugh of those attendants at a funeral or solemn ceremony where the grief is not too intense. Any foolish stimulus may cause the metabolic rebound. A friend of mine once attended an execution. The morning sun was excluded, the shadow and damp of prison walls were everywhere, the usual crowd of curiosity-mongers was present. Upon regaining the open air and sunlight the major part of the crowd burst out laughing with no other external stimulus than the exuberant sunlight.

The mechanics of laughter would also have to take into account the important influence exercised by the diaphragm, the muscular walls of the stomach and glandular activity in the various degrees of the laugh.

Sully mentions the scratching of the head during a state of mental irritation as a well-known instance of the transference of expressive movement from one state of feeling to another, à la Darwin and Wundt. Lauder Brunton explains this habit of the English rustic and similar ones, such as pulling the mustache or beard, or the German habit of slapping the side of the nose with the finger, as a stimulation of some branches of the fifth nerve, thereby causing local dilatation of the cerebral vessels and an added ability to carry on a line of thought. In a similar manner the gustatory branches and the buccal branches of the fifth nerve are stimulated by taking something that has a strong taste, such as brandied cherries. In rural regions peppermint candy is the open sesame of wakefulness in this line. Sucking and chewing and sipping are stimulants greatly increasing the flow of blood through the carotids, as has been determined by experiment. Certain elements of the smoking habit have their *raison d'etre* in activity of this sort. The habit many boys have of spitting on their hands and then of rubbing them together before taking a leap is based on the fact that thereby they obtain a sensori-muscular stimulation. Many mental and bodily automatisms usually explained by reference to some general principles such as inhibition ought to be reëxamined with the view of ascertaining the special causation in question.

Some theories die hard. Of no topic in psychology is this more true than in that of the psychology of the comic. In Sully's 'Essay on Laughter,' and in the article on this subject by Hall and the present writer,[1] may be found *in extenso* a collection of such metaphysical hard-ridden and hard pressed definitions. Nor are Sully and other modern writers altogether free from blame in this respect. Miss Calkins in her 'Introduction to Psychology' says that 'virtually all theories of the comic agree in defining the sense of humor as enjoyment of an unessential incongruity' (p. 284). Sully says, "The most promising way of bringing the several laughable qualities and aspects of things under one descriptive head would seem to be to say that they all illustrate a presentation of something in the nature of a defect, a failure to satisfy some standard requirement, as that of law or custom, provided that it is small enough to be viewed as a harmless plaything" (p. 139).

It is a Ptolemaic pastime trying to discover the causes and inner essence of laughter in the objective world, or even for that matter in the world of mental presentations. In the treatment of the emotions no scientific grounds for causal explanation or classification can be found in the objects of the emotions; no more can such be found for laughter, one of the prominent forms of emotion. The real causal ground of laughter is to be found in physiologic processes. A person may laugh when tickled, may laugh from the influence of drugs, may laugh automatically without the presence of mental presentations, may laugh as an exhibition of *bien être*, may laugh at a button on his coat, may laugh when there is only one single presentation in the mental field or when there are two or more. Morever, these external things are not laughable in themselves. It is our reaction which clothes them with the cloak of humor, gayety, or what-not. In this the comic follows the general law of all emotions, including also under that term the field of æsthetics. These emotional judgments are revelations and judgments of our own selves and characters, rather than of the mountains, sculpture, paintings, or so-called laughable

[1] Hall and Allin, The Psychology of Tickling, Laughing, and the Comic, Amer. Journal of Psychology, Vol. IX, No. 1.

things. Thus the cockney's account of his exploit at a fire (quoted from the London *Times*) would hardly be yours or mine: "'Jump, yer silly fool!' we shouted, 'we've got a sheet!' and he did jump, and there weren't no bloomin' sheet, and he broke 'is bloomin' neck. Larf! I thought I sh'd 'ave died o' larfin'." That which is high tragedy to the gods in the gallery may be comedy to the parquet, and *vice versâ*. "Avast thou wretch!" cries the demi-mondaine actress, "Far rather would I wear the filthy rags of poverty than don the imperial robes of sin." The artistic part of your nature laughs while your moral nature is full of pity; meanwhile there is joy in the 'nigger heaven' over another sinner repulsed.

Evidently the causal element lies in vaso-motor and nervous processes. The sense of joy present in the feeling of *bien être*, in the witticism, in the mild atmosphere of humor, is evidently due to vaso-motor phenomena and a discharge of surplus-stored energy where the discharge does not involve too much strain, effort or lesion. The laughter as a motor phenomenon may continue automatically, finally producing lesion and pain and in some cases death. In the more highly evolved form of this process, such as in wit, the element of suddenness is paramount, brought about by the coalescing of nervous currents seldom or never associated and by sudden vaso-motor and metabolic changes. In other words, we are dealing ultimately with mild forms of vaso-motor shock. Thus Dr. Edward E. Hale was taken when a boy to hear his father speak on a critical occasion. He was so impressed by hearing the orator cry; "Will any man dare say * * *" that he shouted from the gallery, "No, pa!" Neither of these elements taken by themselves are laugh-producing, neither can the ideas by themselves produce such a result, but the vaso-motor shock and sudden coalescence of nervous currents may excite by association the motor centers to intense activity.

In other words, it is not an appeal to our sense of superiority, to our feeling for the ludicrous, to this feeling or to that; the enjoyment we call humor or wit is the result of vaso-motor and nervous changes. The objects of the humor or wit may be numberless, or rather co-extensive with one's experience, but the fundamental or underlying

process will be the same. The concept incongruity may therefore be interpreted with more propriety as the unusual. These unusual coalitions of wit and laughter, however, may at times be eminently fitting or congruous.

The laughter induced by nitrogen monoxide or by cannabis indica is probably hyperæmic or congestive in its origin. The *Rausch* in all its forms, æsthetical, political, religious, spirituous, etc., ought also to be treated in this connection. Some psychologist with Atwater courage ought to make a study of the possible individual and social utility of the *Rausch*. The savages, it is well known, induced this intoxication by various means.

Walter E. Roth, in describing certain songs of the northern Australian aborigines,[1] relates an interesting fact concerning the genesis of savage emotion. He says that, "while the songs are in progress, one, two or more men—any that like—will take into their mouths, chew and spit out again, the leaves of the 'stinging tree' (*Lapartea* sp.). What with the pain and irritation so produced, such an individual is speedily aroused into a state bordering on frenzy, when he will commence eating the human excreta prepared for the purpose, will both act and give expression to anything foul and bestial he can think of, do his best to insult everybody present, start chasing the women, and, rushing hither and thither, will finally fall to the ground completely exhausted and collapsed. The mental and physical pain to which the person is thus subjected may be gauged from the fact that it requires some few weeks before he is sufficiently recovered to resume his ordinary routine of daily duties.

A most pernicious doctrine rather prevalent in theories of æsthetics and play is that of self-illusion. One author even goes so far as to say, 'Make-believe, pretence, representation, are of the essence of play, mirth, and art.' It is a case where theory and half-baked analysis run blindly against the facts. The pretence or self-illusion is in the majority of cases quite as illusive as the grin of the Cheshire cat. Sully says 'play is free activity entered upon for its own sake'

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(1) Walter E. Roth, B. A., M. R. C. S., etc., in Bulletin No. 4, 'North Queensland Ethnography, Games, Sports and Amusements,' Brisbane, March, 1902, p. 22.

(p. 146). 'Play ceases to be pure play just as soon as the end, for example conquest, begins to be regarded as a thing of consequence to the player' (p. 147). Karl Groos also makes much of this theory, saying, for instance, that 'joy in conquest' is the end of play combats ('Play of animals,' pp. 291–2).

I do not deny that there are some play-activities into which there enter pretence and make-believe, but it may be remarked at the same time that such plays are very poor play. In studying the phenomena of play two standpoints must be strictly observed, namely, the subjective and the objective. Subjectively the player, if he plays in earnest, that is to say if it is the best type of play, resembles closely in his activity the so-called serious occupations of life. If it were not so it would not be a useful training for after-life. Play is in many ways modeled after social life, and is *the* social life for the child. It is desperately real to him, and he wonders often why adults are living such a miserable, artificial life, making money and spending wearisome laborious days for ends which are hardly worth the while. The boy who 'monkeys' or 'fools' at practice games is warned off the field by the coach. No pretence or make-believe is wanted. They play and play to win something outside the play-impulse itself. My psychology students tell me that when they play on the football field all their psychological knowledge about play being a preparation for life drops away from them and they play to win their way to the goal line. It is a serious but withal a joyous occupation to them. Such psychological knowledge may injure to some extent the complete engrossment in these preparatory occupations. The true player drops the word preparatory and simply believes these activities to be serious occupations in which he is tremendously interested. The same is true of plays of a more youthful age. The doll, for the time, is a student in school, is sick, naughty, etc. The tin horse with fore legs longer than the hind legs has longer legs; these crass adults who talk differently are talking of another world of horses. The myths and legends of the child's world are very real worlds to him.

Objectively, of course, we look upon the activities of these early stages of growth as preparatory. To call it a world of pretence is to apply a misnomer and to judge poorly of the value of play. Adults are subject to selection, so also are these preliminary stages, but it ought to be called a propædeutical selection, one by the way not yet recognized by biologists, psychologists or sociologists.[1]

Miss McCracken,[2] in speaking of the poverty-stricken girls of the working classes of a certain city, says very aptly:

"In the first place they have gone to the theatre, and they go to the theatre to see the play; not the players, nor to see how they play the play, nor why they possibly play it thus, nor why they do not play it in some other way ('in any conceivable other way,' as I overheard a critic murmur at a recent Shakespearean revival), nor what the author of the play meant, nor what he did not mean, nor what he should have meant. They may see all these things; they frequently do see several of them; but they go to the theatre to see the play. It is interesting to remember that in Shakespeare's time the entire audience went to see the play."

Moreover, the only true criterion of play is the performance of an activity with ease and mastery and with the spirit of pleasure. All else is work or indifferently work or play. If this thesis is granted,[3] then play must not be confined to what we may call traditional forms of play, but must be extended even to adult occupations when performed with the spirit of pleasure and with ease and mastery. For these reasons laughter may be classed as a form of play.

One more point only in this discussion. H. M. Stanley[4] and Sully[5] suggest that teasing may well be taken as the starting point

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(1) A further discussion of this topic will appear shortly under the title of 'Propædeutical Selection.' See also the writer's article on 'Play' in the *University of Colorado Studies*, Vol. 1, No. 1, and Mr. H. A. Carr's paper on 'The Survival Values of Play' in the *Investigations of the Department of Psychology and Education of the University of Colorado*, Vol. I, No. 2.

(2) Elizabeth McCracken, 'The Play and the Gallery,' *Atlantic Manthly*, April, 1902.

(3) See articles quoted above.

(4) H. M. Stanley, discussion of paper by Hall and Allin on 'Tickling, Laughter and the Comic, etc.,' *Psychological Review*, 1899, p. 87.

(5) Sully, p. 84.

in the evolution of play. Taking merely traditional forms of play into account, this *a priori* statement seems to be hardly warranted by anthropological data. The hypothesis seems to underlie this statement that play is a single impulse, a faculty-of-the-mind affair, whereas it is simply protean in its concrete forms. But leaving this point aside, we can safely lay claim to some actual historical data. Buecher, in his 'Arbeit und Rythmus,'[1] clearly proves that many songs, dances, and early forms of literature had their origin in the work activities of early men. It is needless here to repeat the evidence adduced to prove the assertion. Then, again, many ancestral adult activities have been modified to suit childish needs; many present-day adult activities are modified in the same way. But farther back than all this we may go and say that play entered into those species in which parental care began to shield their plastic young from the incidence of natural selection. Then propædeutical selection entered, whereby the preliminary, introductory, educative activities and occupations suitable to the particular species in question survived, building and moulding for the larger life of the adult. That joy accompanied such a process we can reasonably believe, taking as an analogy the exuberance and fullness of life of youth wherever we find it.

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(1) Followed, and to some extent extended by Gummere, "The Beginnings of Poetry."

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